Karlsruhe Institute of Technology

High Performance Humanoid Technologies

Robotik I: Introduction to Robotics

Chapter 2 – Kinematics

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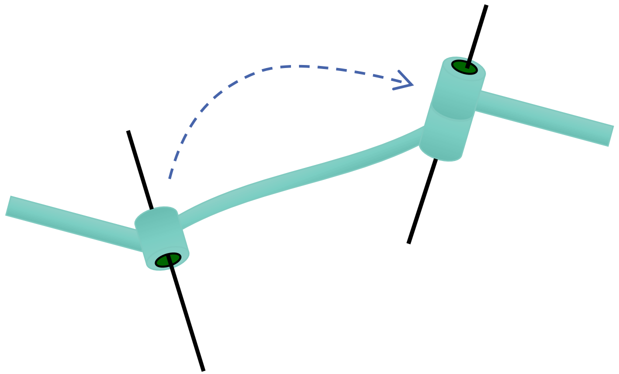
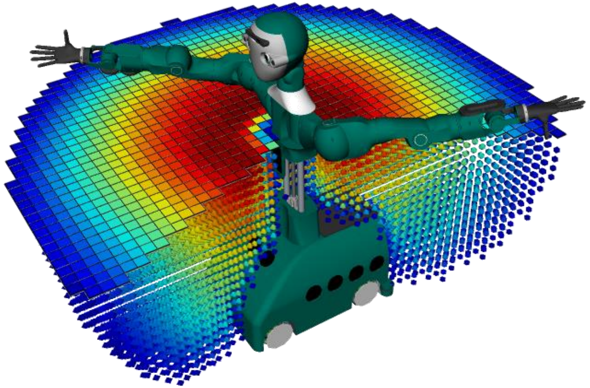
https://www.humanoids.kit.edu





Models in Robotics

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- **Kinematic Models**
  Kinematics studies of motion of bodies and systems based **only on geometry**, i.e. without considering the physical properties and the forces acting on them. The essential concept is a **pose** (position and orientation).

- **Dynamic Models**
  Dynamics studies the relationship between the **forces and moments** acting on a robot and accelerations they produce,

- **Geometric Models**
  Geometry: Mathematical description of the **shape of bodies**

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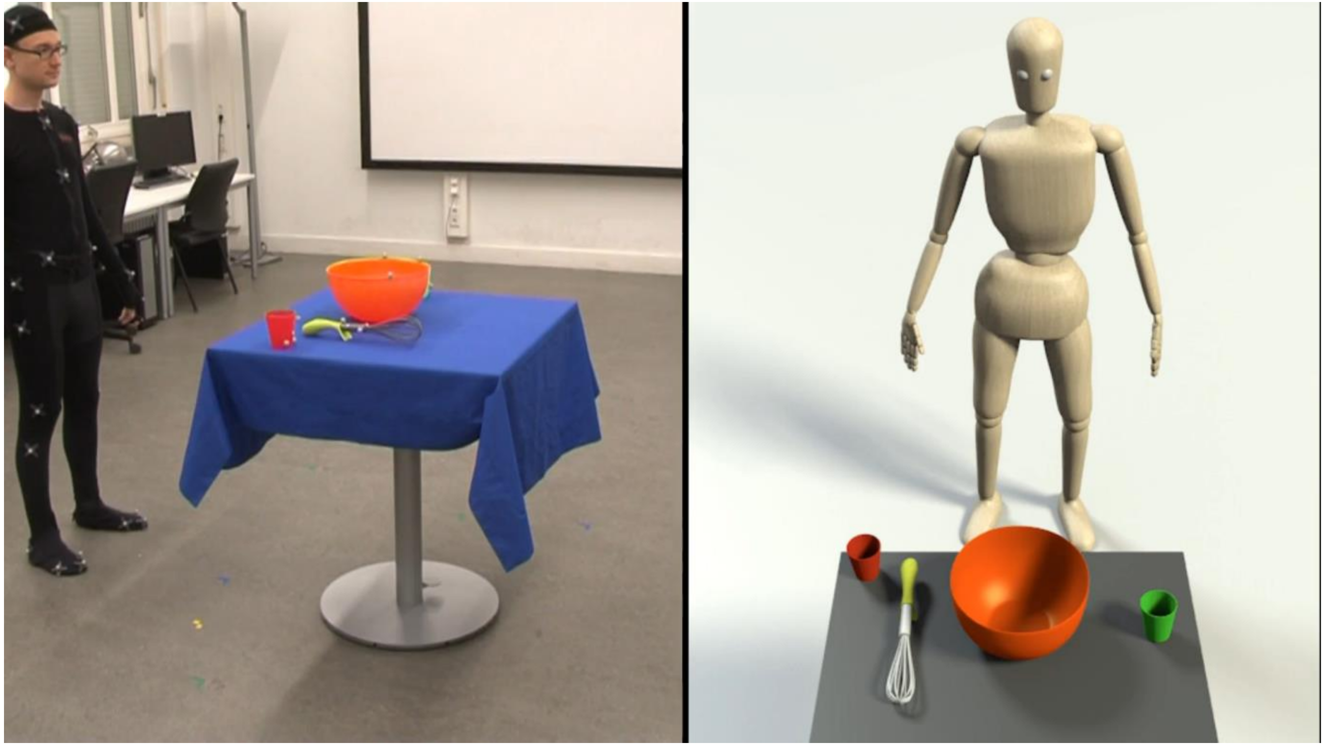
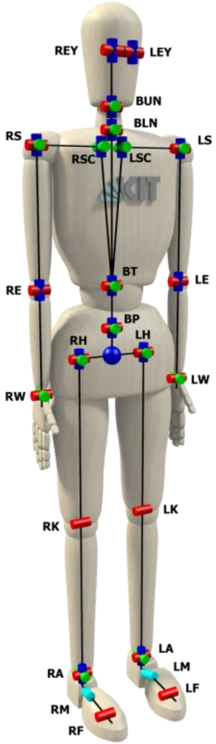
- **Kinematic Model**
  - Kinematic Chain
  - Denavit-Hartenberg Convention
  - Direct Kinematics Problem
  - Examples
  - Jacobian Matrix
  - Singularities and Manipulability
  - Representation of Reachability
- Geometric Model
  - Areas of Application
  - Classification
  - Examples

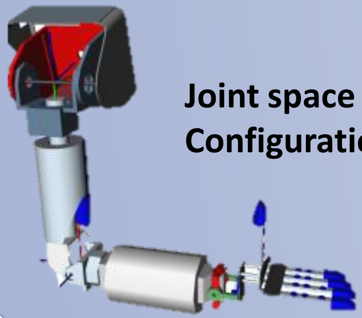
Kinematic Model (1)





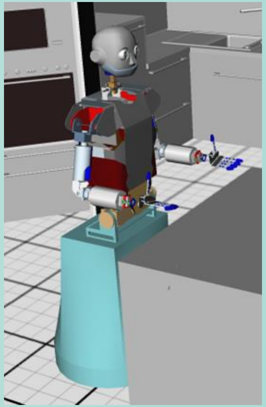
Kinematic Model (2)

Two spaces



Joint space / Configuration space (C)

Task space / Workspace (W)



Kinematic Model (3)

- Definition
  - The **kinematic model** of a robot describes the relationships between the **joint space** (robot coordinates, configuration space) and the **space of end effector poses** in world coordinates (task space, Cartesian space).

- Areas of application
  - Relationship between **joint angles** and **poses** of the end effector
  - **Reachability analysis**
  - Geometric relation between the **body parts of the robot** (self-collision)
  - Geometric relation to the **environment** (collision detection)

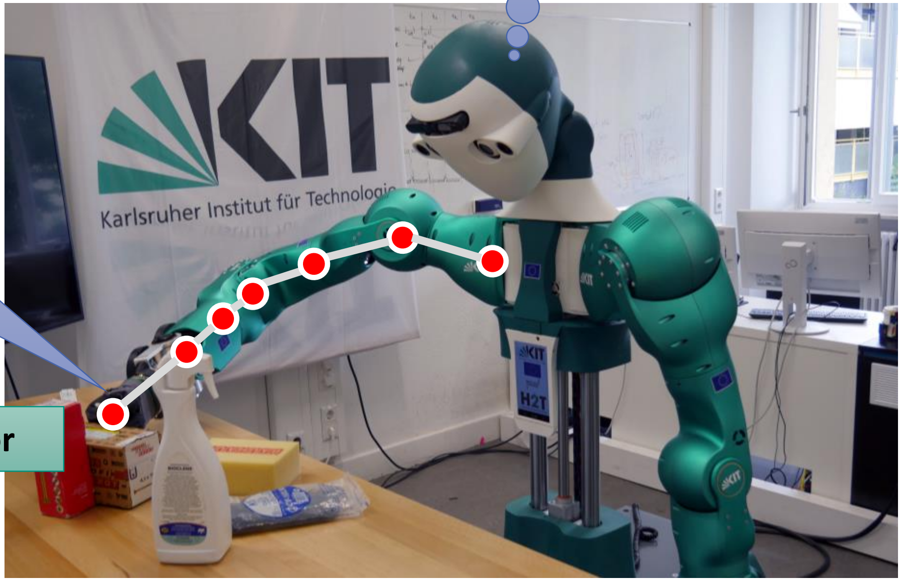
Forward Kinematics

- **Direct** kinematics problem
  - Input: Joint angles of the robot
  - Output: **Pose of the end effector**

Where is my hand?

Forward Kinematics: HERE!

End effector

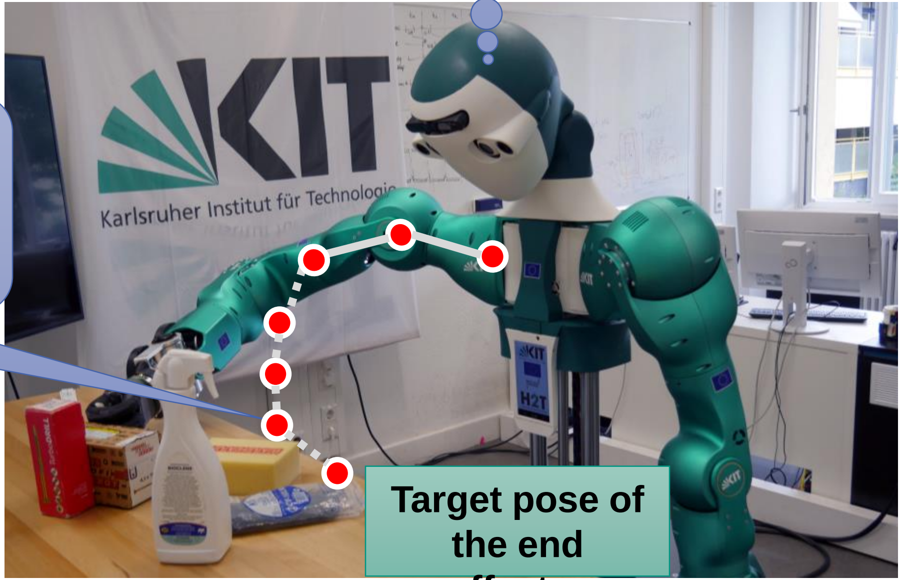


Inverse Kinematics

- **Inverse** kinematics problem
  - Input: Target pose of the end effector
  - Output: **Joint angles**

How do I move my hand to the target?

Inverse Kinematics: Determines the joint angles



Outline: Direct and Inverse Kinematics

Joint space (configuration space)

Transformation

Cartesian coordinates (task space)

Direct Kinematics

$$\boldsymbol{x} = f(\boldsymbol{\theta})$$

$$(\theta_1, \dots, \theta_n) \in \boldsymbol{C} \subseteq \mathbb{R}^n$$

$$\boldsymbol{X} \subset \mathbb{R}^m$$

Inverse Kinematics

$$\boldsymbol{\theta} = f^{-1}(\boldsymbol{x})$$

Position and orientation of the end effector

$$\boldsymbol{x}_{EEF} = (x, y, z, \alpha, \beta, \gamma)$$

n: Joint degrees of freedom (DoF)
m: Cartesian degrees of freedom

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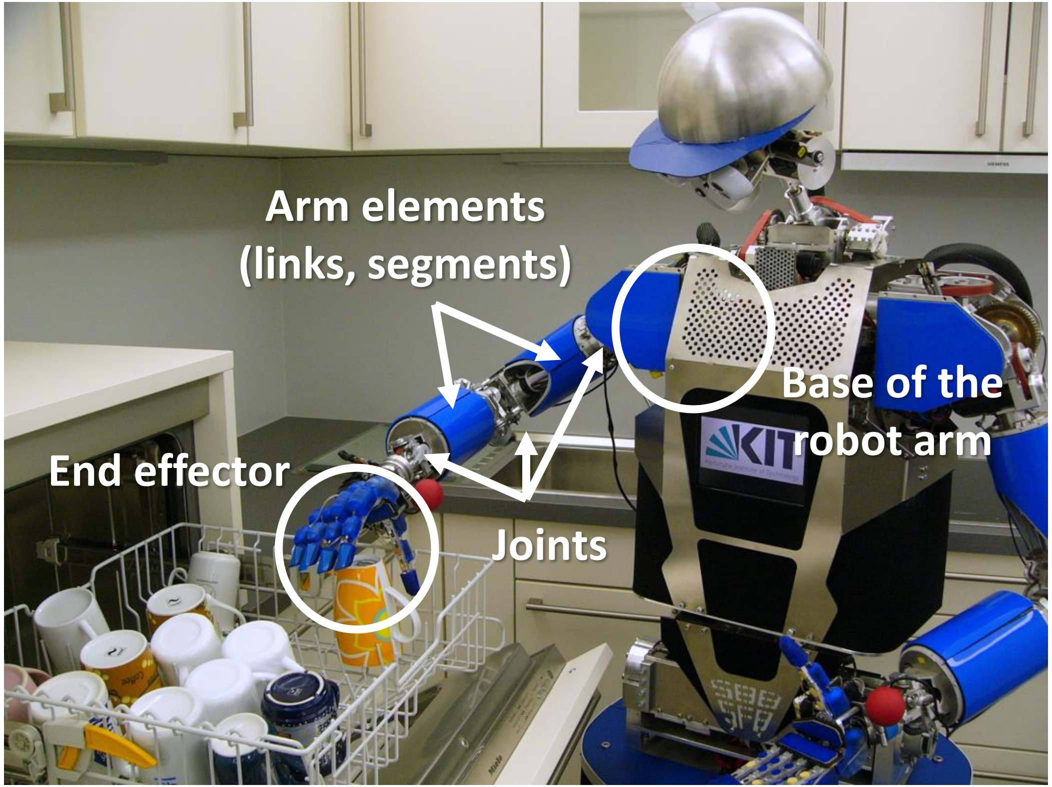
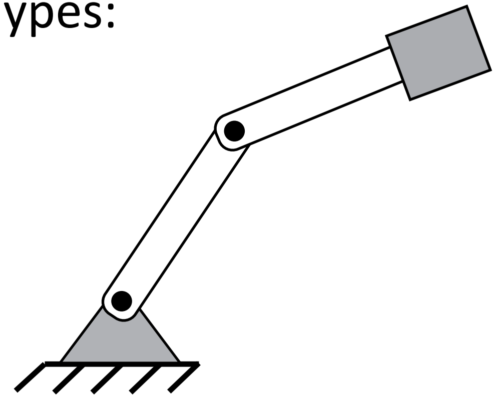
Elements of a Kinematic Chain



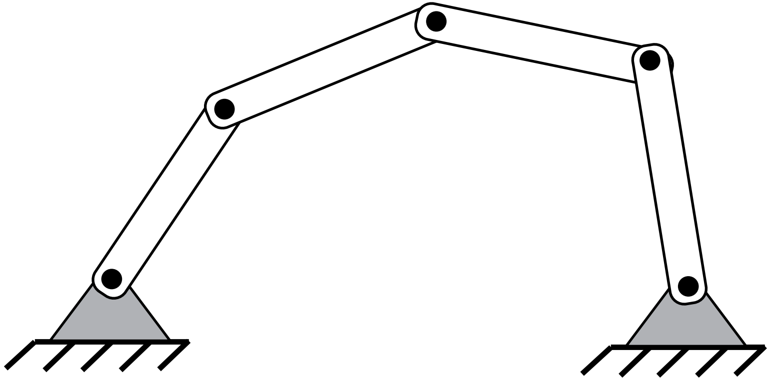
Kinematic Chain: Definition

- Definition:
  A kinematic chain is formed by **several bodies** that are **kinematically connected by joints** (e.g. robot arm).

- Types:



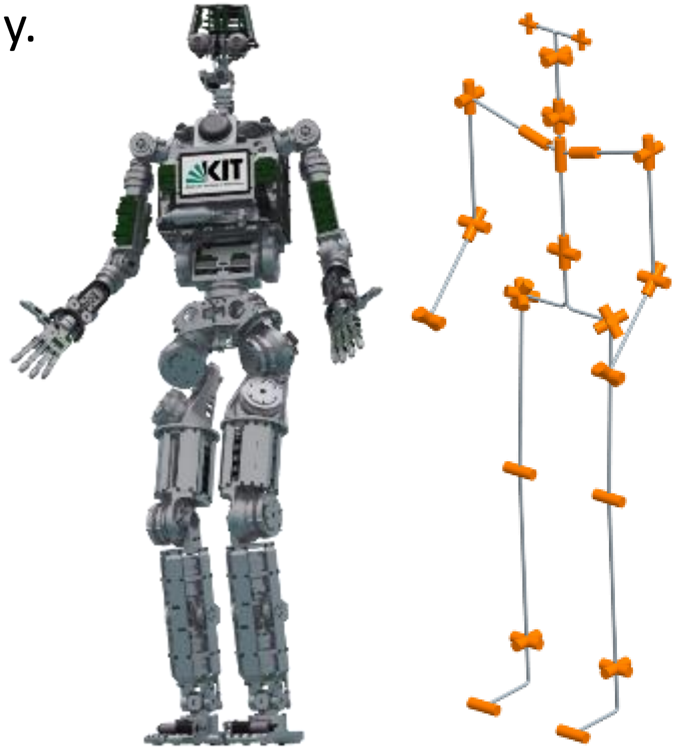
Open kinematic chain



Closed kinematic chain

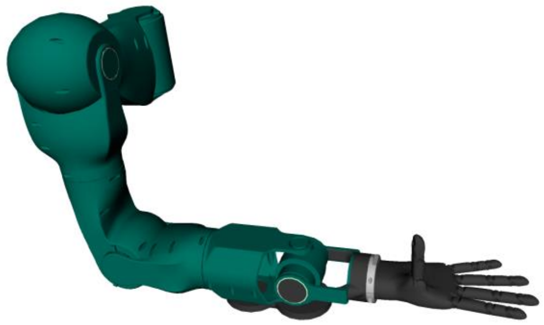
Kinematic Chain: Conventions

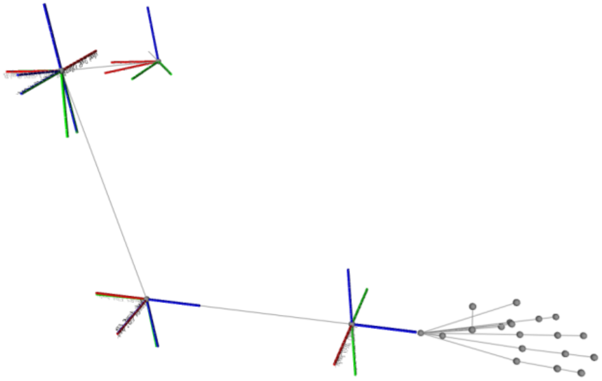
- Each **arm element** corresponds to one **rigid** body.
- Each arm element is connected to the next one by a **joint**.
- For prismatic and rotational joints:
  Each **joint** has only **one degree of freedom** (translation or rotation).
- **Kinematic parameters:**
  - Joint definition (e.g. rotation axis)
  - Transformation between joints



Kinematic Parameters

- **Joint parameters**
  - Revolute joint: rotation axis
  - Prismatic joint: direction of translation
  - …
- Specification of the **positions of joints relative to each other**
  - Fixed transformation between two joints
  - Defines the local coordinate systems of the joints
  - Transformation from joint $i-1$ to joint i with transformation matrix ${}^{i-1}T_i$





Number of Parameters of the Kinematic Chain

- A **transformation** must be determined for each link:
  - **3 rotation parameters**
  - **3 translation parameters**

➔ **6 parameters** per link of the kinematic chain

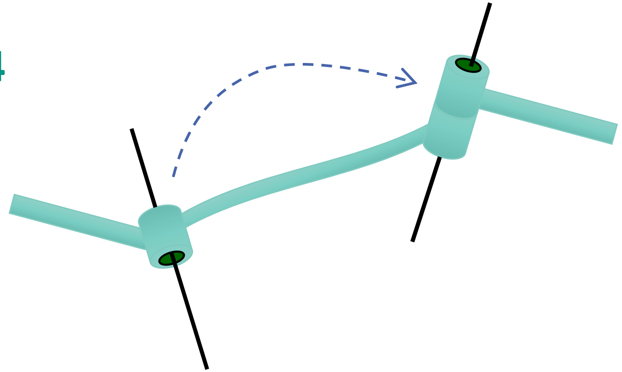
Contents

Denavit-Hartenberg (DH) Convention

- Goal: **Reduction of the parameters** for describing an arm element

- Properties
  - Systematic description of relations (**translations** and **rotations**) **between adjacent joints**
  - Reduction of the number of **parameters from 6 to 4**

- Description with homogeneous matrices



Literature: Denavit, Hartenberg: „A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices“, Journal of Applied Mechanics, 1955, vol 77, pp 215-221

DH Convention for the Choice of Coordinate Systems

- Each coordinate system is determined on the basis of the following three rules:
  1. The z_{i-1}**-axis** lies along the **axis of movement** of the $\boldsymbol{i}$**-th joint**
  2. The x_i**-axis** lies along the **common normal** of z_{i-1} and z_i
     (direction via cross product: $x_i = z_{i-1} \times z_i$)
  3. The y_i**-axis** completes the coordinate system according to the **right-hand rule**

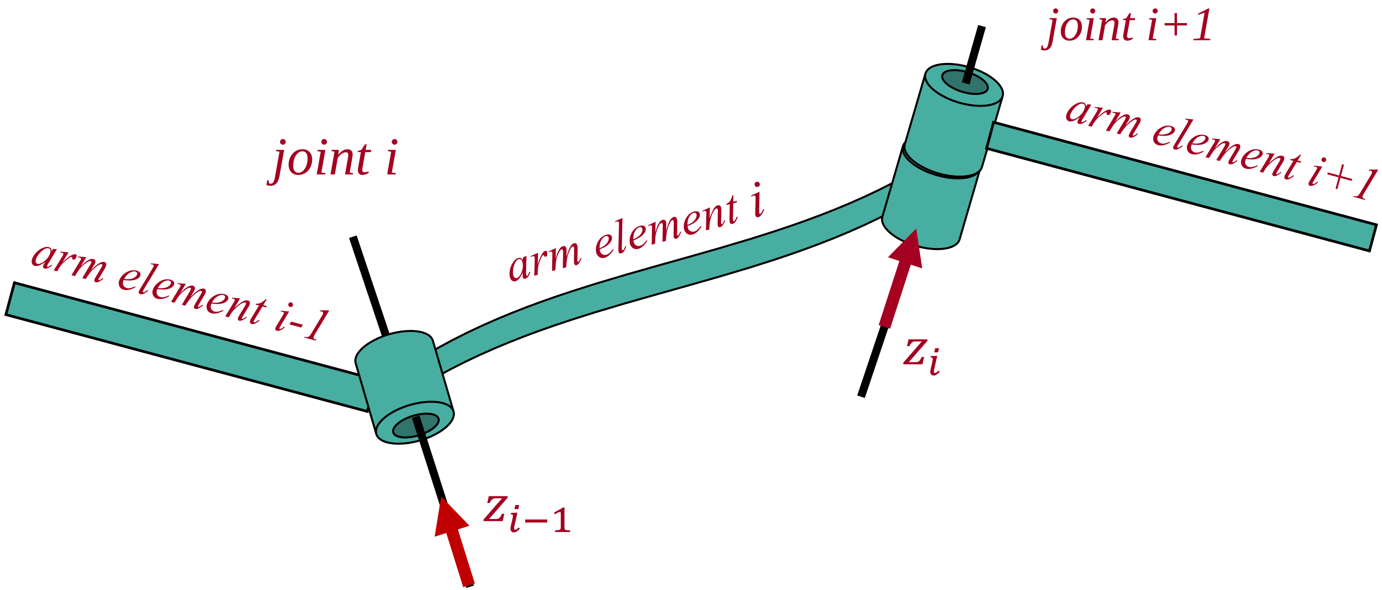
$$i \in \{\text{base}, 1, \dots, n\}$$

→Derivation of parameters for arm element and joint

- **Remark**
  - Other variants of the DH convention can also be found in the literature
  - In this lecture we consider the modified variant of Waldron and Paul

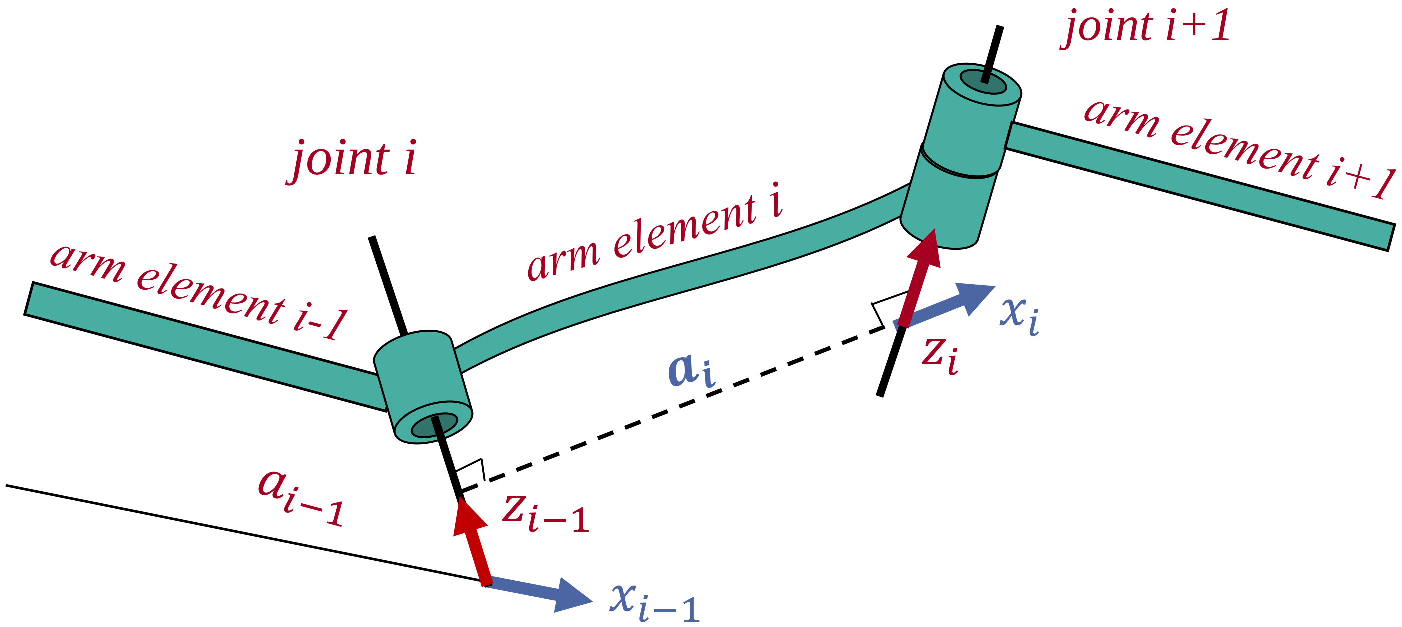
DH Convention: Parameters of the Arm Element (1)

- Each **arm element** i is embedded between two **joints** i and $i+1$
- z_i runs along the **joint axis** $i+1$



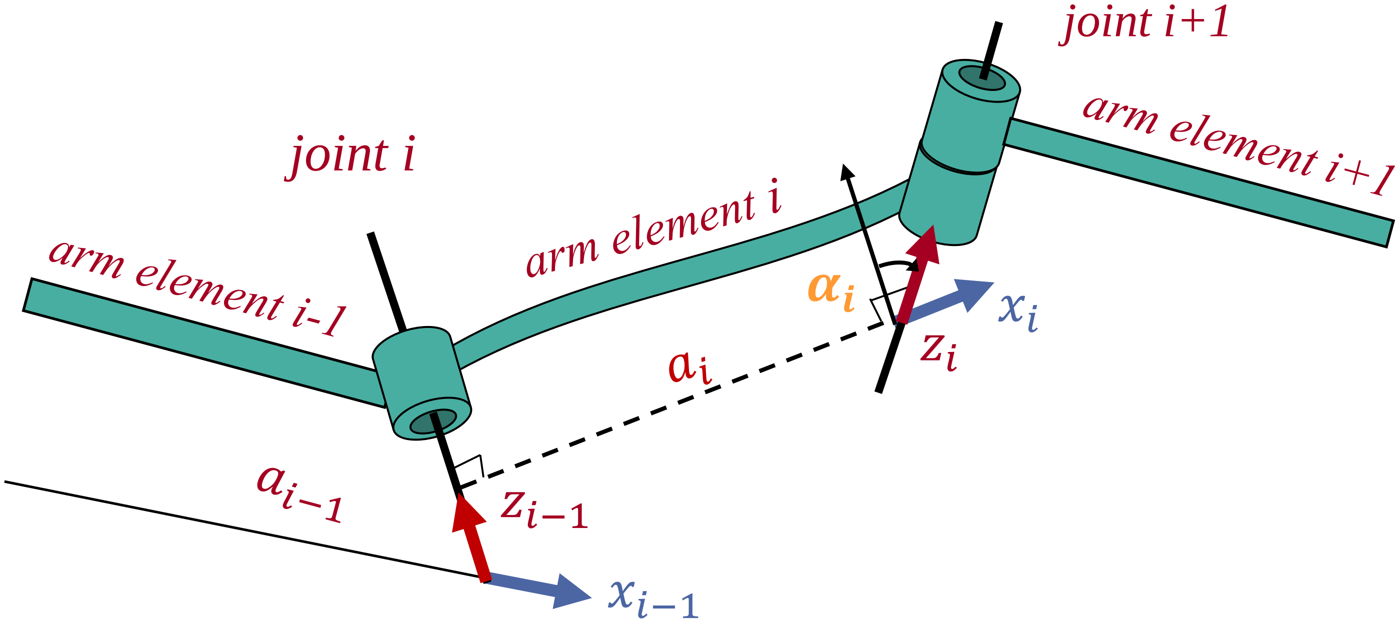
DH Convention: Parameters of the Arm Element (2)

- **Link length** a_i of an arm element i describes the **distance** from z_{i-1} to z_i
- x_i lies along the **normal of** z_{i-1} **and** z_i (cross product)



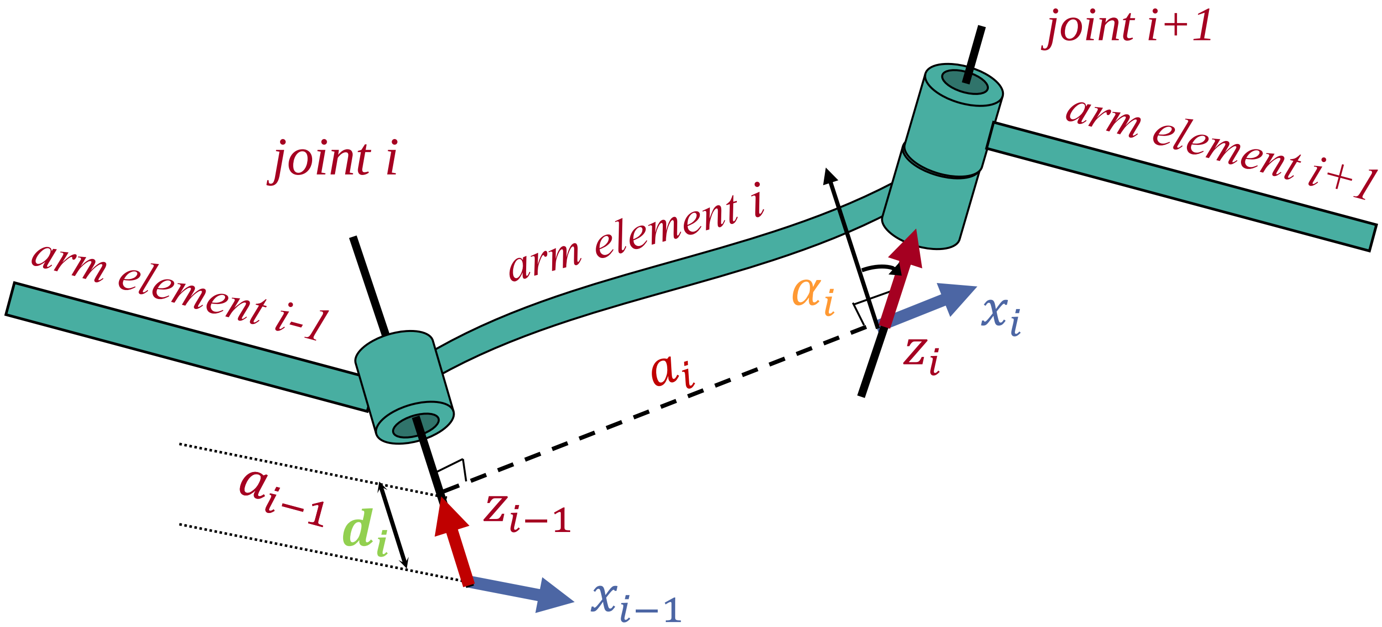
DH Convention: Parameters of the Arm Element (3)

- **Link twist α_i** describes the **angle** from z_{i-1} to z_i **around x_i**.



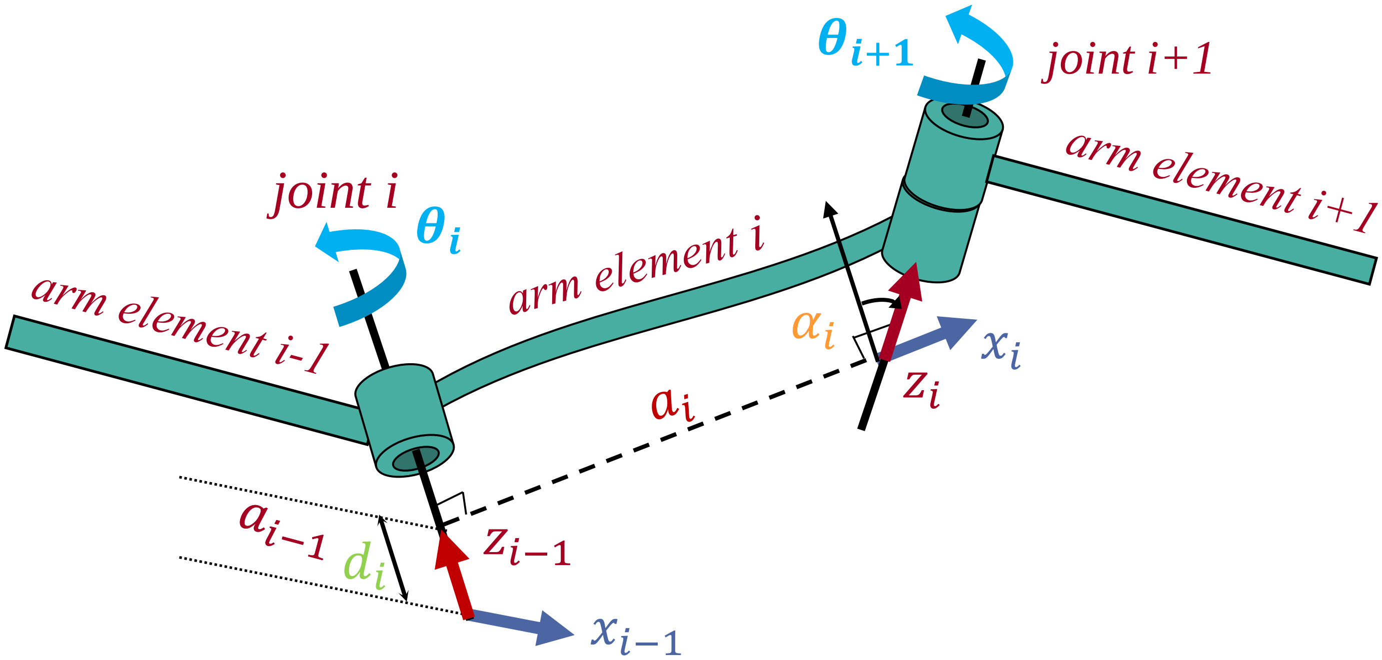
DH Convention: Parameters of the Arm Element (4)

- **Link offset d_i** is the distance between x_{i-1}-axis and x_i -axis **along the z_{i-1}**-axis



DH Convention: Parameters of the Arm Element (5)

- **Joint angle θ_i** is the angle from x_{i-1} to x_i **around z_{i-1}**



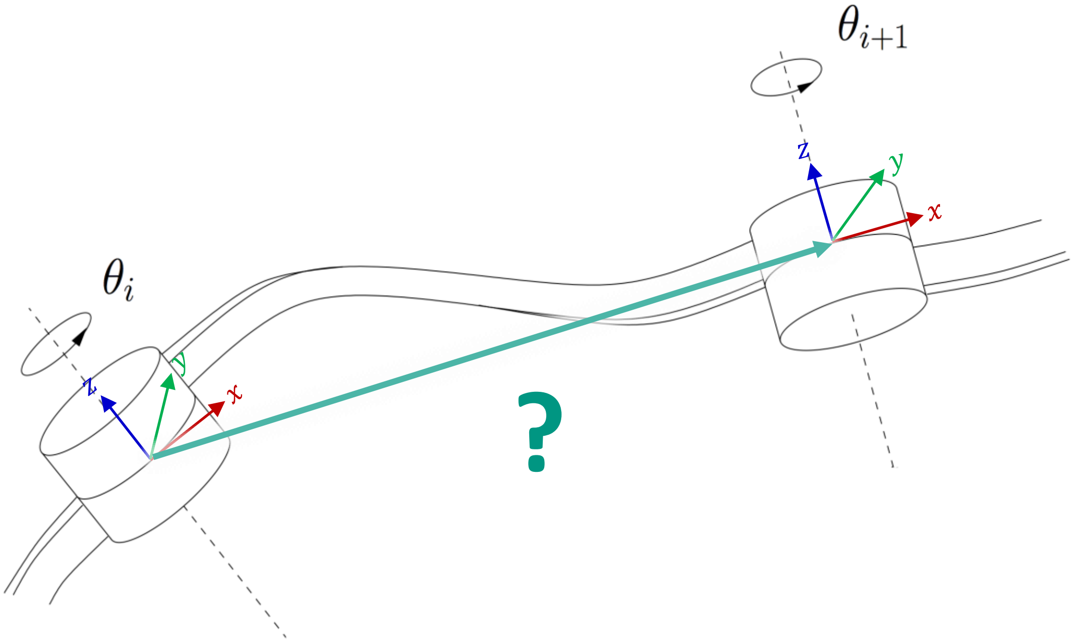
DH Parameters

- The **four parameters** a_i , α_i , d_i and θ_i are called **DH parameters**.
- They describe the transformations between two successive rotational or translational robot joints

DH Parameters (Denavit-Hartenberg Parameters)

Parameter	Symbol	Revolute joint	Prismatic joint
Link length	a	constant	constant
Link twist	α	constant	constant
Link offset	d	constant	**variable**
Joint angle	θ	**variable**	constant

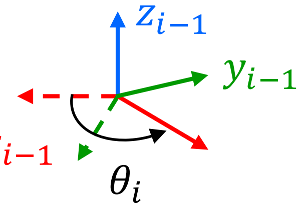
Transformation Between Two Robot Joints



DH Transformation Matrices (1)

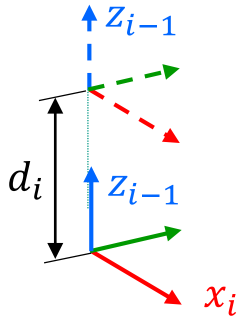
Transformation from LCS_{i-1} to LCS_i

1. A **rotation** θ_i around the z_{i-1}**-axis** so that the x_{i-1}-axis is parallel to the x_i-axis.



$$R_{z_{i-1}}(\theta_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. A **translation** d_i along the z_{i-1}**-axis** to the point where z_{i-1} and x_i intersect.



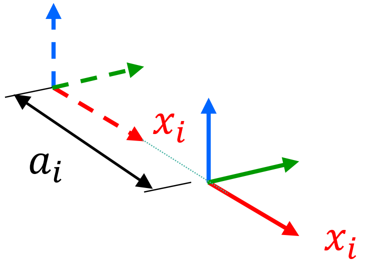
$$T_{z_{i-1}}(d_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LCS_i: *Local Coordinate System of joint i*

DH Transformation Matrices (2)

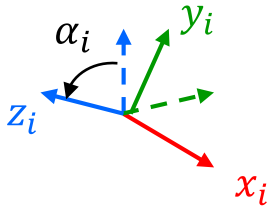
Transformation from LCS_{i-1} to LCS_i

3. A **translation** a_i along the x_i**-axis** to align the origins of the coordinate systems.



$$T_{x_i}(a_i) = \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. A **rotation** α_i around the x_i**-axis** to convert the z_{i-1}-axis into the z_i-axis.



$$R_{x_i}(\alpha_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LCS_i: *Local Coordinate System of joint i*

DH Transformation Matrices (3)

Transformation LCS_{i-1} to LCS_i

$$A_{i-1,i} = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(a_i) \cdot R_{x_i}(\alpha_i)$$
$$= \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse DH Transformation

Transformation from LCS_{i-1} to LCS_i

$$A_{i-1,i}^{-1} = A_{i,i-1}$$

$$= \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\cos\alpha_i \cdot \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & \sin\alpha_i & -d_i \cdot \sin\alpha_i \\ \sin\theta_i \cdot \sin\alpha_i & -\sin\alpha_i \cdot \cos\theta_i & \cos\alpha_i & -d_i \cdot \cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{pmatrix} n_x & o_x & a_x & u_x \\ n_y & o_y & a_y & u_y \\ n_z & o_z & a_z & u_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T^{-1} = \begin{pmatrix} n_x & n_y & n_z & -n^T\mathbf{u} \\ o_x & o_y & o_z & -o^T\mathbf{u} \\ a_x & a_y & a_z & -a^T\mathbf{u} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

See chapter 1

Concatenation of DH Transformations

- By concatenating the DH matrices, the pose of individual **coordinate systems relative** to the **reference coordinate system** can be determined.

- Position of the m-th coordinate system relative to the base coordinate system:

$$S_{\text{base},m}(\boldsymbol{\theta}) = A_{0,1}(\theta_1) \cdot A_{1,2}(\theta_2) \cdot \ldots \cdot A_{m-2,m-1}(\theta_{m-1}) \cdot A_{m-1,m}(\theta_m)$$
$$= \begin{pmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

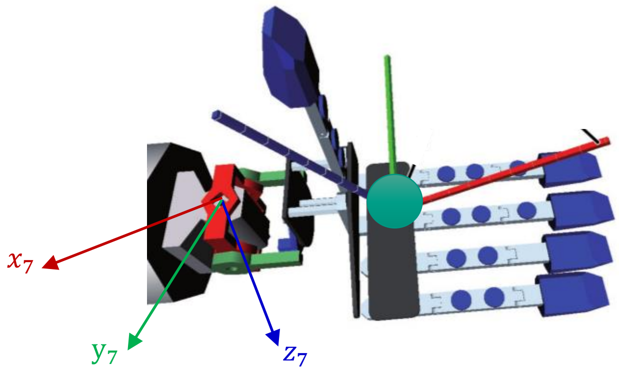
- This is a mapping of the configuration space $\mathrm{C} \subset \mathbb{R}^n$ to the workspace $\mathrm{W} \subset \mathbb{R}^m$

$$\mathbb{R}^n \rightarrow \mathbb{R}^m: \qquad \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta})$$

DH Parameters – Notes

- The **four parameters** a_i , α_i , d_i and θ_i are called **DH parameters**.

- **Important:** Reference coordinate system (RCS) and end effector coordinate system (ECS) of the kinematic chain
  - As intuitive as possible; set so that the associated DH parameters are simple (preferably zero)
  - RCS as the coordinate system of the first joint in zero position
  - End effector coordinate system at an ‘important reference point’ at the end effector



Summary: Determination of the DH Parameters

1. **Sketch** of the manipulator
2. Identify and **enumerate** the **joints** (1, …, last link = n)
3. Draw the **axes** z_{i-1} for **each joint** $\boldsymbol{i}$
4. Determine the **parameters** a_i between $z_{i-1}$and z_i
5. Draw the x_i**–axes**
6. Determine the **parameters** α_i (twist around the x_i-axes)
7. Determine the **parameters** d_i (link offset)
8. Determine the **angles** θ_i around the z_{i-1}-axes
9. Compose the **joint transformation matrices** $A_{i-1,i}$

Contents

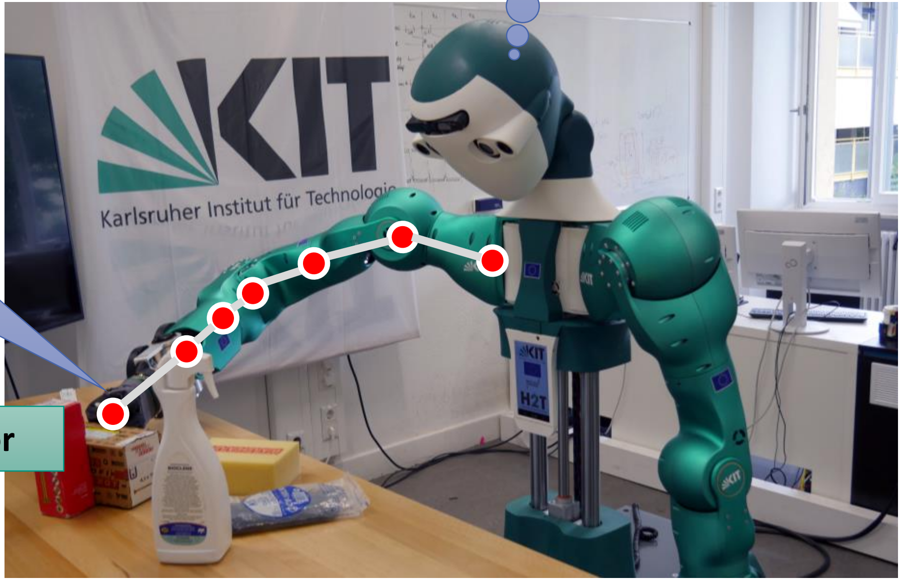
Direct Kinematics Problem (1)

- **Direct** kinematics problem
  - Input: Joint angles of the robot
  - Output: **Pose of the end effector**

Where is my hand?

Forward Kinematics: HERE!

End effector



Direct Kinematics Problem (2)

- The pose of the end effector (EEF) is to be determined from the DH parameters and the joint angles.

- The pose of the end effector (EEF) in relation to the RCS is given by:

$$S_{RCS,EEF}(\theta) = A_{0,1}(\theta_1) \cdot A_{1,2}(\theta_2) \cdot \ldots \cdot A_{n-2,n-1}(\theta_{n-1}) \cdot A_{n-1,n}(\theta_n)$$

$$= \begin{pmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

- Joint angles $\theta_1, \ldots, \theta_n$ are given $\Rightarrow$ **The pose of the EEF is obtained from the equation above by inserting the joint angle values.**

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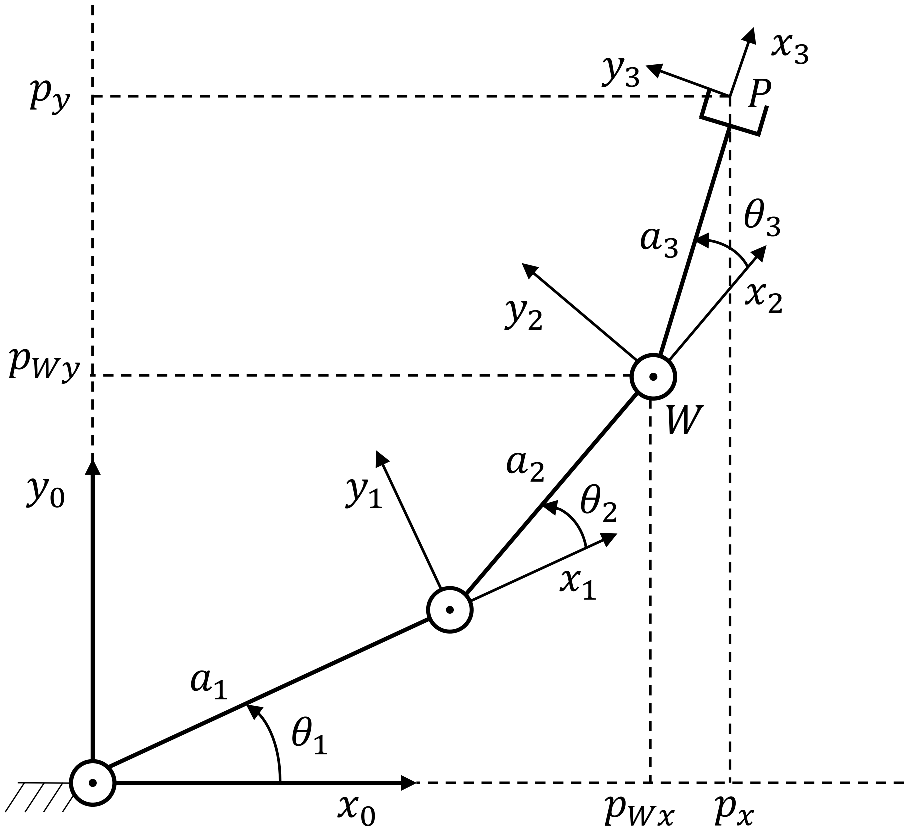
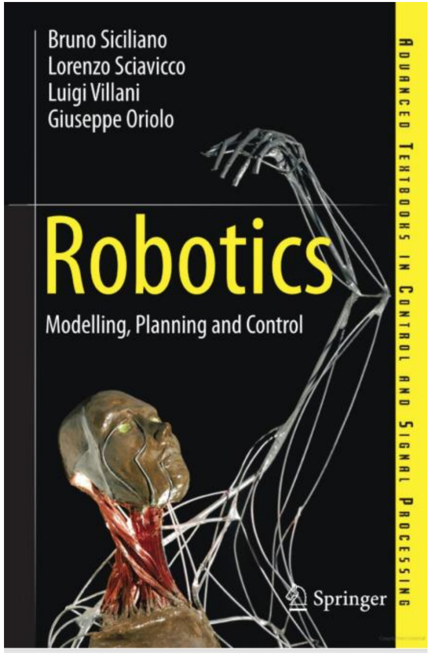
Example 1: Planar Robot (in xy-Plane)

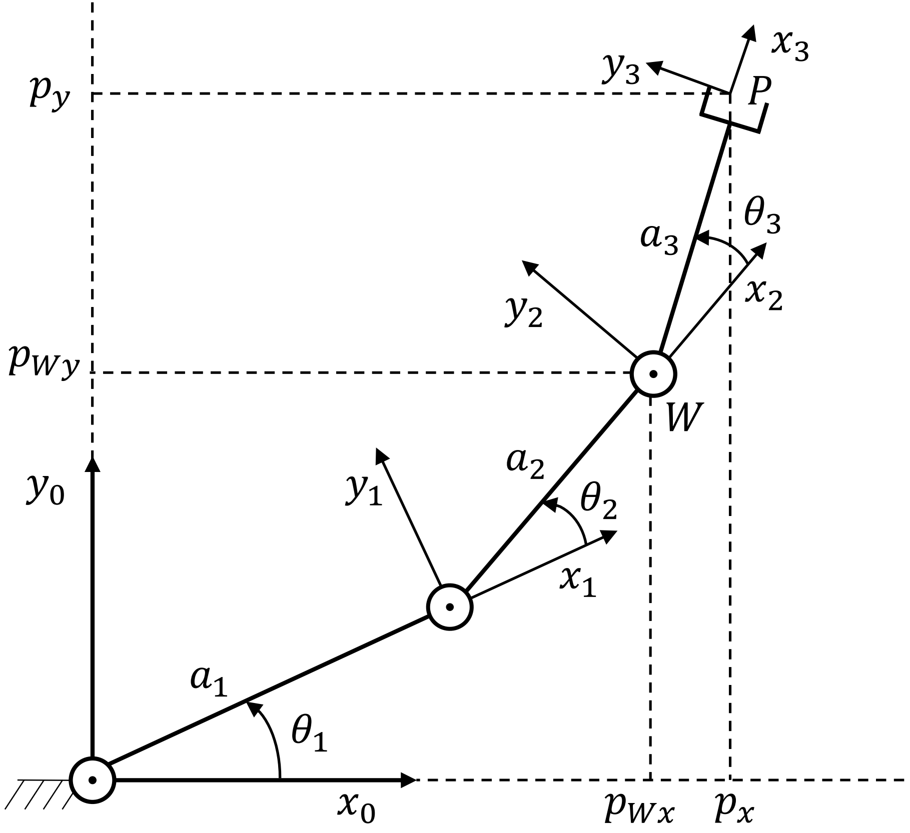




Example 1: Planar Robot

z-axes are parallel

No translation in z-direction



Joint	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$A_{i-1,i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

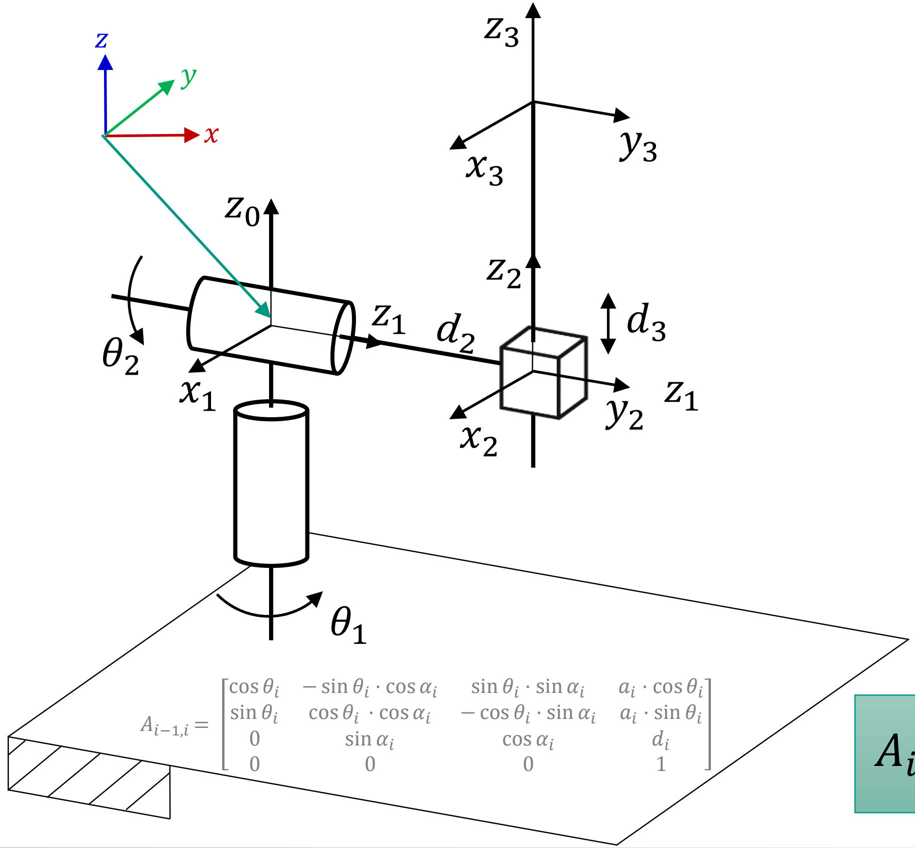
$$A_{i-1,i} = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 1: Planar Robot

$$A_{0,3}(\theta) = A_{0,1} \cdot A_{1,2} \cdot A_{2,3} = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Abbreviations: $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, $s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$, etc.

Example 2: 3D Robot



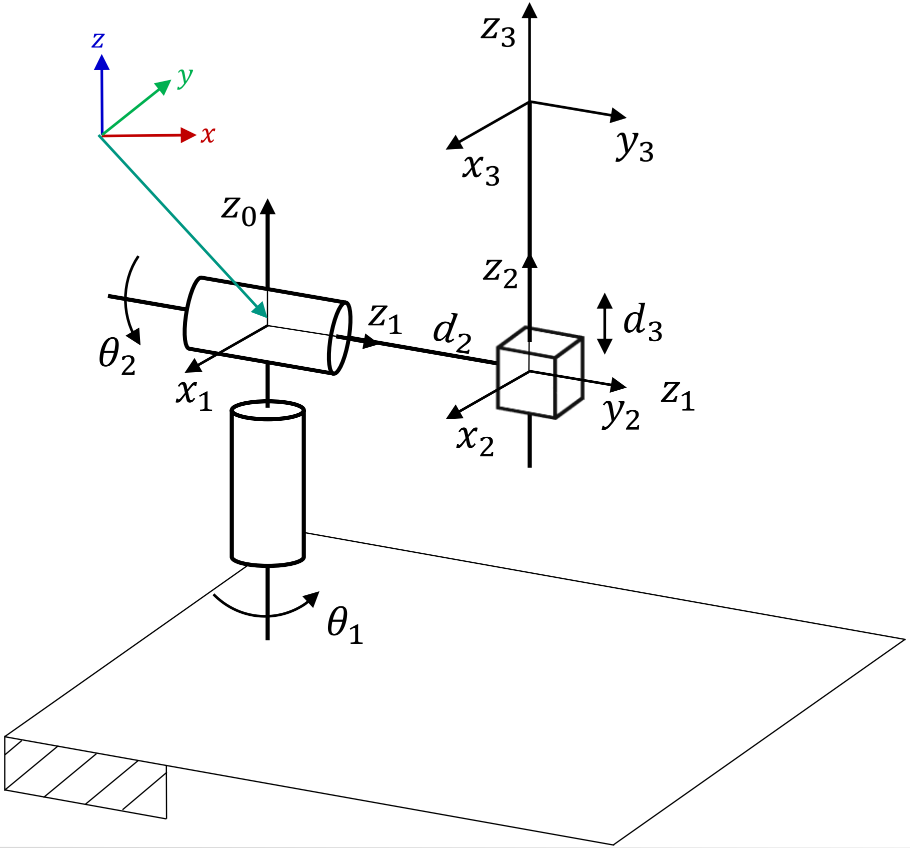
$$A_{i-1,i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Joint	a_i	α_i	d_i	θ_i
1	0	-90	0	θ_1
2	0	90	d_2	θ_2
3	0	0	d_3	0

$$A_{0,1} = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{i-1,i} = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(a_i) \cdot R_{x_i}(\alpha_i)$$

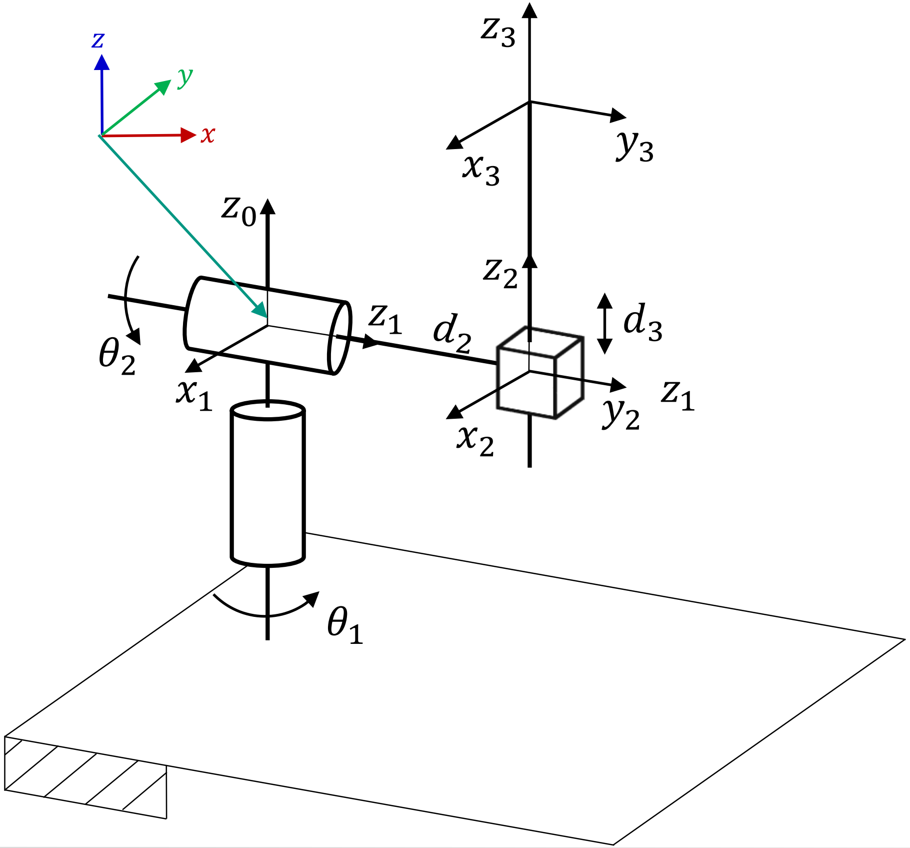
Example 2: 3D Robot



Joint	a_i	α_i	d_i	θ_i
1	0	−90	0	θ_1
2	0	90	d_2	θ_2
3	0	0	d_3	0

$$A_{1,2} = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2: 3D Robot



Joint	a_i	α_i	d_i	θ_i
1	0	-90	0	θ_1
2	0	90	d_2	θ_2
3	0	0	d_3	0

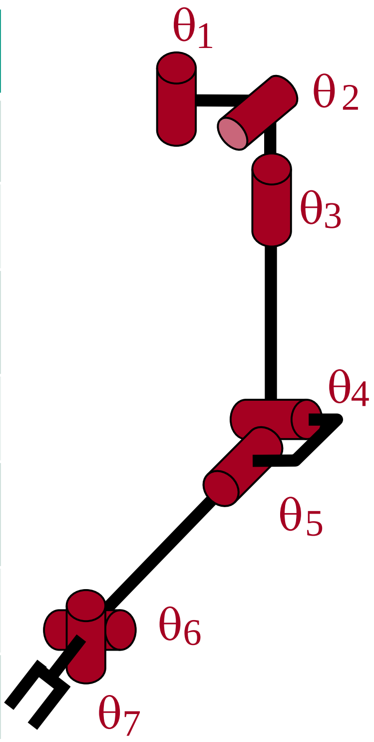
$$A_{2,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2: 3D Robot

$$A_{0,3}\ (\theta) = A_{0,1} \cdot A_{1,2} \cdot A_{2,3} = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 d_3 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 s_2 d_3 + c_1 d_2 \\ -s_2 & 0 & c_2 & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH Notation: Arm of ARMAR-I

Joint i	θ_i [°]	a_i [mm]	α_i [°]	d_i [mm]
1	θ_1	30	-90	0
2	$\theta_2 - 90$	0	-90	0
3	$\theta_3 + 90$	0	90	223,5
4	θ_4	0	-90	0
5	θ_5	0	90	270
6	$\theta_6 + 90$	0	-90	0
7	θ_7	140	90	0



Forward Kinematics (1)

- **Given:** $\theta,\ A_{i-1,i}(\theta)$
- **Desired:** $S_{base,EEF}(\theta)$

$$A_{i-1,i}(\theta) = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(a_i) \cdot R_{x_i}(\alpha_i)$$

$$= \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_{base,EEF}(\theta) = A_{0,1}(\theta_1) \cdot A_{1,2}(\theta_2) \cdot \ldots \cdot A_{n-2,n-1}(\theta_{n-1}) \cdot A_{n-1,n}(\theta_n)$$

$$= \begin{pmatrix} R & \boldsymbol{t} \\ \boldsymbol{0}^{\boldsymbol{T}} & 1 \end{pmatrix}$$

Forward Kinematics (2)

- Pose of the end effector coordinate system relative to the base:

$$S_{base,EEF}(\boldsymbol{\theta}) = A_{0,1}(\theta_1) \cdot A_{1,2}(\theta_2) \cdot \ldots \cdot A_{n-2,n-1}(\theta_{n-1}) \cdot A_{n-1,n}(\theta_n)$$

- This is a mapping of the configuration space $\mathrm{C} \subset \mathbb{R}^n$ to the workspace $\mathrm{W} \subset \mathbb{R}^m$

$$\mathbb{R}^n \rightarrow \mathbb{R}^m: \qquad \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\theta})$$

Derivation of the Forward Kinematics

- Forward Kinematics: **Joint angle position → end effector pose**

$$\mathbb{R}^n \rightarrow \mathbb{R}^m: \quad \boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{\theta}(t))$$

Pose of the EEF in W — Joint angle vector in C

- How do the corresponding relationships look like?
  - Joint angular velocities → end effector velocities
  - Joint torques → end effector forces and torques

- **Approach**: Derive differential forward kinematics → **Jacobian matrix**

Contents

- Kinematic Model
  - Kinematic Chain
  - Denavit-Hartenberg Convention
  - Direct Kinematics Problem
  - Examples
  - **Jacobian Matrix**
  - Singularities and Manipulability
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- Geometric Model
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Reminder: Jacobian Matrix

- Given a differentiable function

$$f\colon \mathbb{R}^n \to \mathbb{R}^m \quad \left\{ \text{i.e. } \boldsymbol{f}(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \right.$$

- The **Jacobian Matrix** contains all first-order partial derivatives of f.
  For $\boldsymbol{a} \in \mathbb{R}^n$:

$$J_f(\boldsymbol{a}) = \left(\frac{\partial f_i}{\partial x_j}(\boldsymbol{a}) \right)_{i,j} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\boldsymbol{a}) \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$f_1, \ldots, f_m$ denote the component functions of f and $x_1, \ldots, x_n$ the coordinates in $\mathbb{R}^n$.

Jacobian Matrix in Forward Kinematics

- **Problem**: Forward kinematics is matrix-valued (n: number of joints)

$$f\colon \mathbb{R}^n \rightarrow SE(3)$$

$\Rightarrow$ Jacobian matrix not defined

- **Solution**: Select vector representation, e.g. use roll, pitch and yaw angles to represent orientations

$$f\colon \mathbb{R}^n \rightarrow \mathbb{R}^6 \left\{ \begin{pmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{pmatrix} \right.$$

End Effector Velocities

- **Assumption:** The kinematic chain moves along a trajectory

$$\theta : \mathbb{R} \rightarrow \mathbb{R}^n$$

- Pose of the end effector $\boldsymbol{x}(t) \in \mathbb{R}^6$ at time t:

$$\boldsymbol{x}(t) = f(\theta(t))$$

- The end effector velocity depends linearly on the joint velocities (**chain rule**):

$$\dot{\boldsymbol{x}}(t) = \frac{\partial f(\theta(t))}{\partial t} = \frac{\partial f(\theta(t))}{\partial \theta} \cdot \frac{\partial \theta(t)}{\partial t} = J_f(\theta(t)) \cdot \dot{\theta}(t)$$

End Effector Velocities

- The Jacobian matrix relates Cartesian end effector velocities to joint angle velocities

$$\dot{\boldsymbol{x}}(t) = J_f\big(\theta(t)\big) \cdot \dot{\theta}(t)$$

- The following problems can be solved with this relation:
  - **Forward kinematics in the velocity space**:
    Given joint angle velocities,
    which Cartesian end effector velocities result from them?
  - **Inverse kinematics in the velocity space**:
    Given Cartesian end effector velocities,
    which joint angle velocities are needed to realize them?

Kinematics using the Jacobian Matrix (1)

Forward kinematics:

Given the joint angle velocities $\dot{\theta}(t)$,
which Cartesian end effector velocities $\dot{\boldsymbol{x}}(t)$ result from them?

Insert $\dot{\theta}(t)$:

$$\dot{\boldsymbol{x}}(t) = J_f\big(\theta(t)\big) \cdot \dot{\theta}(t)$$

Kinematics using the Jacobian Matrix (2)

Inverse kinematics:
Given a Cartesian end effector velocities $\dot{\boldsymbol{x}}(t)$,
which joint angle velocities $\dot{\boldsymbol{\theta}}(t)$ are necessary to realize them?

$$\dot{\boldsymbol{x}}(t) = J_f\big(\theta(t)\big) \cdot \dot{\boldsymbol{\theta}}(t)$$

$$\Big\downarrow \quad J_f^{-1}\big(\theta(t)\big) \cdot [\quad]$$

$$\dot{\boldsymbol{\theta}}(t) = J_f^{-1}\big(\theta(t)\big) \cdot \dot{\boldsymbol{x}}(t)$$

Forces and Torques at the End Effector

- **Assumption:** The kinematic chain moves along a trajectory

$$\theta: \mathbb{R} \rightarrow \mathbb{R}^n$$

- The **work** done (force × distance) must remain constant regardless of the reference system (**friction neglected**)

$$\int_{t_1}^{t_2} \dot{\theta}(t)^T \cdot \tau(t)\, dt = W = \int_{t_1}^{t_2} \dot{x}(t)^T \cdot F(t)\, dt$$

- With:

$\dot{\theta}(t)\colon \mathbb{R} \rightarrow \mathbb{R}^n,$	Joint velocities
$\tau(t)\colon \mathbb{R} \rightarrow \mathbb{R}^n,$	Joint torques
$\dot{x}(t)\colon \mathbb{R} \rightarrow \mathbb{R}^6,$	End effector velocities
$F(t)\colon \mathbb{R} \rightarrow \mathbb{R}^6,$	Force-torque vector at the end effector

Forces and Torques at the End Effector

$$\int_{t_1}^{t_2} \dot{\theta}(t)^T \cdot \tau(t)\, dt = W = \int_{t_1}^{t_2} \dot{x}(t)^T \cdot F(t) dt$$

- The relation must apply for each time interval $[t_1, t_2]$, therefore:

$$\dot{\theta}(t)^T \cdot \tau(t) = \dot{x}(t)^T \cdot F(t)$$

- Known relation between end effector velocity and Jacobian matrix:

$$\dot{\theta}(t)^T \cdot \tau(t) = \dot{\theta}(t)^T \cdot J_f^T(\theta(t)) \cdot F(t)$$

$$\dot{\boldsymbol{x}}(t) = J_f(\theta(t)) \cdot \dot{\boldsymbol{\theta}}(t)$$

- Since $\dot{\theta}(t)$ is arbitrary, it follows that:

$$\tau(t) = J_f^T(\theta(t)) \cdot F(t)$$

Forces and Torques at the End Effector

- The Jacobian matrix relates forces and torques at the end effector to the torques in the joints:

$$\tau(t) = J_f^T\big(\theta(t)\big) \cdot F(t)$$

- The following problems can be solved with this relation:
  - Given forces/torques at the end effector, which torques must act in the joints to resist them?
  - Given the torques in the joints, which resulting forces and torques act on the (fixed) end-effector?

Recap – DH Transformation Matrices

Transformation from LCS_{i-1} to LCS_i

$$
\begin{aligned}
A_{i-1,i} &= R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(a_i) \cdot R_{x_i}(\alpha_i) \\
&= \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Recap – Direct and Inverse Kinematics

Joint angle space (configuration space)

Transformation

Cartesian coordinates (workspace)

Direct Kinematics

$$\boldsymbol{x} = f(\boldsymbol{\theta})$$

$$(\theta_1, \dots, \theta_n) \in \boldsymbol{C} \subseteq \mathbb{R}^n$$

$$\boldsymbol{X} \subset \mathbb{R}^m$$

Inverse Kinematics

$$\boldsymbol{\theta} = f^{-1}(\boldsymbol{x})$$

E.g. position and orientation of the end effector

$$\boldsymbol{x}_{EEF} = (x, y, z, \alpha, \beta, \gamma)$$

n: Joint degrees of freedom (DoF)
m: Cartesian degrees of freedom

Recap: Forward Kinematics in Position and Velocity Space

Position space: $\boldsymbol{x}(t) = f\left(\theta(t)\right)$

$$\dot{\boldsymbol{x}}(t) = \frac{\partial f(\theta(t))}{\partial t} = \frac{\partial f(\theta(t))}{\partial \theta} \cdot \frac{\partial \theta(t)}{\partial t} = J_f\left(\theta(t)\right) \cdot \dot{\theta}(t)$$

Velocity space: $\dot{\boldsymbol{x}}(t) = J_f\left(\theta(t)\right) \cdot \dot{\theta}(t)$

Recap: Inverse Kinematics

$$\dot{\boldsymbol{x}}(t) = J_f\big(\theta(t)\big) \cdot \dot{\boldsymbol{\theta}}(t)$$

$$J_f^{-1}\big(\theta(t)\big) \cdot [\quad]$$

$$\dot{\boldsymbol{\theta}}(t) = J_f^{-1}\big(\theta(t)\big) \cdot \dot{\boldsymbol{x}}(t)$$

Recap – Jacobian Matrix

- Velocity space

$$\dot{\boldsymbol{x}}(t) = J_f\big(\theta(t)\big) \cdot \dot{\theta}(t)$$

- Force space

$$\tau(t) = J_f^T\big(\theta(t)\big) \cdot F(t)$$

Recap – Jacobian Matrix

$$\boldsymbol{x} = f(\theta)$$

$$\dot{\boldsymbol{x}} = J_f(\theta) \cdot \dot{\boldsymbol{\theta}}$$

$$J_f(\boldsymbol{\theta}) = \left(\frac{\partial f_i}{\partial \theta_j}(\boldsymbol{\theta})\right)_{i,j} = \begin{pmatrix} \frac{\partial f_1}{\partial \theta_1}(\boldsymbol{\theta}) & \cdots & \frac{\partial f_1}{\partial \theta_n}(\boldsymbol{\theta}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial \theta_1}(\boldsymbol{\theta}) & \cdots & \frac{\partial f_m}{\partial \theta_n}(\boldsymbol{\theta}) \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$\boldsymbol{x} = (x, y, z, \alpha, \beta, \gamma)^T \in \mathbb{R}^{m=6}$ and $\theta \in \mathbb{R}^n$

$$J_f(\boldsymbol{\theta}) = \left(\frac{\partial f_i}{\partial \theta_j}(\boldsymbol{\theta})\right)_{i,j} = \begin{pmatrix} \frac{\partial x}{\partial \theta_1}(\boldsymbol{\theta}) & \cdots & \frac{\partial x}{\partial \theta_n}(\boldsymbol{\theta}) \\ \frac{\partial y}{\partial \theta_1}(\boldsymbol{\theta}) & & \frac{\partial y}{\partial \theta_n}(\boldsymbol{\theta}) \\ \vdots & \ddots & \vdots \\ \frac{\partial \gamma}{\partial \theta_1}(\boldsymbol{\theta}) & \cdots & \frac{\partial \gamma}{\partial \theta_n}(\boldsymbol{\theta}) \end{pmatrix} \in \mathbb{R}^{6 \times n}$$

Calculation of the Jacobian Matrix

- Each column of the Jacobian matrix corresponds to a joint θ_i of the kinematic chain

$$J_f = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \cdots & \frac{\partial f}{\partial \theta_n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$$J_f(\boldsymbol{\theta}) = \left(\frac{\partial f_i}{\partial x_j}(\boldsymbol{\theta}) \right)_{i,j} = \begin{pmatrix} \frac{\partial f_1}{\partial \theta_1}(\boldsymbol{\theta}) & \cdots & \frac{\partial f_1}{\partial \theta_n}(\boldsymbol{\theta}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial \theta_1}(\boldsymbol{\theta}) & \cdots & \frac{\partial f_m}{\partial \theta_n}(\boldsymbol{\theta}) \end{pmatrix} \in \mathbb{R}^{m \times n}$$

- **Approach:** Numerical calculation of the Jacobian matrix is carried out **column by column** ⇒ joint by joint

Geometric Calculation of the Jacobian Matrix

$$\boldsymbol{x} = f(\theta)$$

$\boldsymbol{x} = (x, y, z, \alpha, \beta, \gamma)^T \in \mathbb{R}^{m=6}$ and $\theta \in \mathbb{R}^{n=6}$

$$\dot{\boldsymbol{x}} = J(\theta) \cdot \dot{\boldsymbol{\theta}}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} & & j_{16} \\ j_{21} & j_{22} & & j_{26} \\ j_{31} & j_{32} & & j_{36} \\ j_{41} & j_{42} & & j_{46} \\ j_{51} & j_{52} & & j_{56} \\ j_{61} & j_{62} & & j_{66} \end{pmatrix} \cdot \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{pmatrix} \Rightarrow \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \left(J_1(\theta), J_2(\theta), \ldots, J_6(\theta)\right) \cdot \dot{\boldsymbol{\theta}}$$

Geometric Calculation of the Jacobian Matrix

Karlsruhe Institute of Technology

1. Case: Prismatic joint

- **Assumption:** The j-th joint performs a translation in direction of the unit vector $\boldsymbol{z}_j \in \mathbb{R}^3$.
- Therefore:

$$J_j(\boldsymbol{\theta}) = \frac{\partial f(\boldsymbol{\theta})}{\partial \theta_j} = \begin{bmatrix} \boldsymbol{z}_j \\ \boldsymbol{0} \end{bmatrix} \in \mathbb{R}^6$$

2. Case: Revolute joint

- **Assumption:** The j-th joint performs a rotation around the axis $\boldsymbol{z}_j \in \mathbb{R}^3$ at the position $\boldsymbol{p}_j \in \mathbb{R}^3$.
- Therefore:

$$J_j(\boldsymbol{\theta}) = \frac{\partial f(\boldsymbol{\theta})}{\partial \theta_j} = \begin{bmatrix} \boldsymbol{z}_j \times (f_p(\boldsymbol{\theta}) - \boldsymbol{p}_j) \\ \boldsymbol{z}_j \end{bmatrix} \in \mathbb{R}^6$$

$f_p(\boldsymbol{\theta})$: position of the end effector

Geometric Calculation of the Jacobian Matrix

2. Case: **Revolute joint**

$$J_j(\boldsymbol{\theta}) = \frac{\partial f(\boldsymbol{\theta})}{\partial \theta_j} = \begin{bmatrix} \boldsymbol{z}_j \times (f_p(\boldsymbol{\theta}) - \boldsymbol{p}_j) \\ \boldsymbol{z}_j \end{bmatrix} \in \mathbb{R}^6$$

- Manipulator with n joints

$$J(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{z}_0 \times (f_p(\boldsymbol{\theta}) - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (f_p(\boldsymbol{\theta}) - \boldsymbol{p}_1) & \dots & \boldsymbol{z}_{n-1} \times (f_p(\boldsymbol{\theta}) - \boldsymbol{p}_{n-1}) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \dots & \boldsymbol{z}_{n-1} \end{bmatrix}$$

Summary: Jacobian Matrix

- Forward kinematics:

$$f: \mathbb{R}^n \to \mathbb{R}^6, \quad f(\boldsymbol{\theta}) = \boldsymbol{x} = (x, y, z, \alpha, \beta, \gamma)$$

- Jacobian matrix:

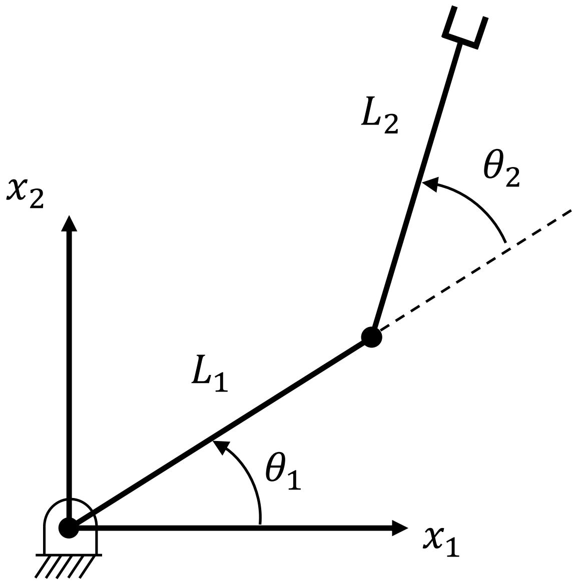
$$J_f = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \cdots & \frac{\partial f}{\partial \theta_n} \end{pmatrix} \in \mathbb{R}^{6 \times n}$$

- Properties:
  - J_f describes the relations between
    - Joint angle velocities (n-dimensional) and end effector velocities (6-dimensional)
    - Joint torques (n-dimensional) and forces and torques at the end effector (6-dimensional)
  - The Jacobian matrix depends on the joint angle configuration

Jacobian Matrix: Example (1)

- Manipulator with two joints θ_1, θ_2

- Find $\dot{\boldsymbol{x}}$



Jacobian Matrix: Example (2)

- Forward kinematics

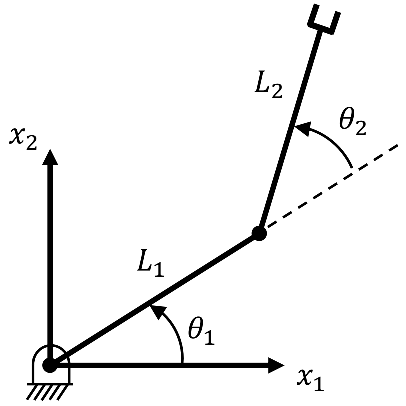
$$\boldsymbol{x} = f(\boldsymbol{\theta})$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

- Velocity of the end effector

$$\dot{\boldsymbol{x}} = J_f(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = J_f(\theta) \cdot \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

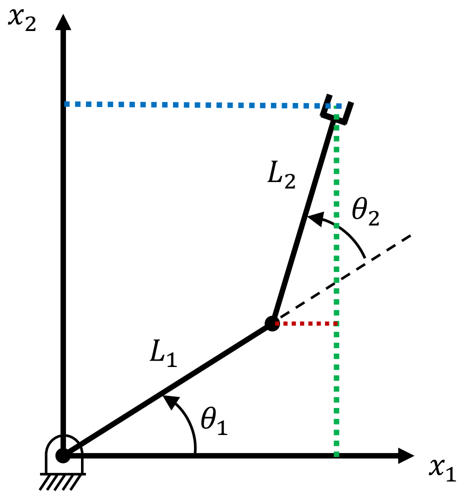


Jacobian Matrix: Example (3)

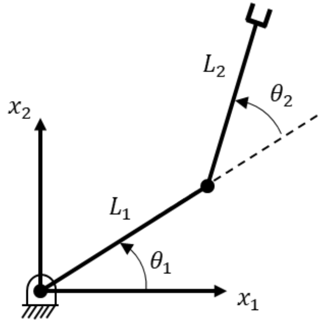
- Forward kinematics

$$x_1 = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$x_2 = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$



Jacobian Matrix: Example (4)



- Forward kinematics

$$x_1 = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$x_2 = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

- Derivation

$$\dot{x}_1 = -L_1\dot{\theta}_1 \sin\theta_1 - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin(\theta_1 + \theta_2)$$
$$\dot{x}_2 = L_1\dot{\theta}_1 \cos\theta_1 + L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos(\theta_1 + \theta_2)$$

- Jacobian matrix

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

$$\underbrace{}_{J_1(\theta)} \qquad \underbrace{}_{J_2(\theta)}$$

Jacobian Matrix: Example (5)

- **End effector velocity**

$$\boldsymbol{v}_{EEF} = J_1(\boldsymbol{\theta})\dot{\theta}_1 + J_2(\boldsymbol{\theta})\dot{\theta}_2$$

- As long as $J_1(\boldsymbol{\theta})$ and $J_2(\boldsymbol{\theta})$ are **not linearly dependent**, a $\boldsymbol{v}_{EEF}$ can be generated in any direction in the $x_1 x_2$-plane.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

- **Singularities**
  - $J_1(\theta)$ and $J_2(\theta)$ linearly dependent $\rightarrow J(\theta)$ singular
  - E.g. if $\theta_2 = 0°$
  - The possible movements of the end effector are restricted.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1) & -L_2 \sin(\theta_1) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1) & L_2 \cos(\theta_1) \end{pmatrix}}_{\begin{pmatrix} -(L_1 + L_2)\sin\theta_1 & -L_2 \sin\theta_1 \\ (L_1 + L_2)\cos\theta_1 & L_2 \cos\theta_1 \end{pmatrix}} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

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  - Examples

Recap – Jacobian Matrix

- Velocity space

$$\dot{\boldsymbol{x}}(t) = J_f\big(\theta(t)\big) \cdot \dot{\theta}(t)$$

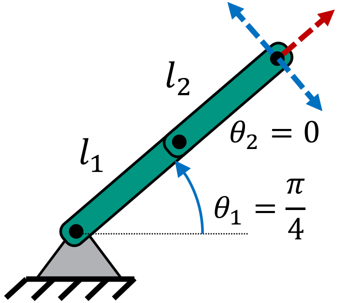
- Force space

$$\tau(t) = J_f^T\big(\theta(t)\big) \cdot F(t)$$

Kinematic Singularities

- If a robot is in a configuration $\boldsymbol{\theta}_{singular} \in C$, in which it is no longer able to move instantaneously in one or more directions, this is referred to as a **kinematic singularity**.
- Configurations $\boldsymbol{\theta}_{singular} \in C$ that lead to a kinematic singularity are called **singular**.
- Can we distinguish singular from non-singular configurations?
  $\rightarrow$ Yes, using the Jacobian matrix

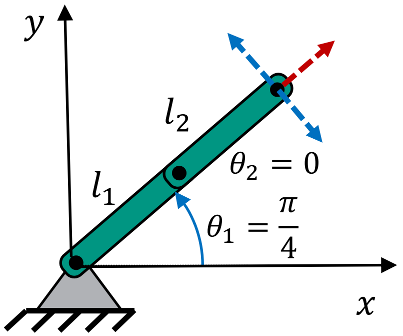
There is no joint angular velocity that generates an end effector velocity in the **red direction**.
$\Rightarrow$ The configuration is singular.



Kinematic Singularities: Example

- Forward kinematics: $\boldsymbol{x} = f(\boldsymbol{\theta}) = \begin{pmatrix} l_1 \cdot \cos\theta_1 + l_2 \cdot \cos(\theta_1 + \theta_2) \\ l_1 \cdot \sin\theta_1 + l_2 \cdot \sin(\theta_1 + \theta_2) \end{pmatrix}$
- Jacobian matrix: $J(\boldsymbol{\theta}) = \begin{pmatrix} -l_1 \cdot \sin\theta_1 - l_2 \cdot \sin(\theta_1 + \theta_2) & -l_2 \cdot \sin(\theta_1 + \theta_2) \\ l_1 \cdot \cos\theta_1 + l_2 \cdot \cos(\theta_1 + \theta_2) & l_2 \cdot \cos(\theta_1 + \theta_2) \end{pmatrix}$
- For the singular configuration $\boldsymbol{\theta} = \left(\frac{\pi}{4}, 0\right)^T$:

$$J\left(\begin{pmatrix} \pi/4 \\ 0 \end{pmatrix}\right) = (\boldsymbol{J}_1, \boldsymbol{J}_2) = \begin{pmatrix} -(l_1 + l_2) \cdot \frac{1}{\sqrt{2}} & -l_2 \cdot \frac{1}{\sqrt{2}} \\ (l_1 + l_2) \cdot \frac{1}{\sqrt{2}} & l_2 \cdot \frac{1}{\sqrt{2}} \end{pmatrix}$$



- The first and second column are linearly dependent

$$\boldsymbol{J}_1 = \frac{l_1 + l_2}{l_2} \cdot \boldsymbol{J}_2$$

Kinematic Singularities: Jacobian Matrix (1)

Forward kinematics in the velocity space:
The end effector velocity is a **linear combination** of the columns of the Jacobian matrix.

$$J(\boldsymbol{\theta}) = \left(\frac{\partial f}{\partial \theta_1}, \quad \frac{\partial f}{\partial \theta_2}, \quad \dots, \quad \frac{\partial f}{\partial \theta_n}\right) = (\boldsymbol{J}_1, \boldsymbol{J}_2 \dots, \boldsymbol{J}_n)$$

$$\dot{\boldsymbol{x}} = J(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{x}} = (\boldsymbol{J}_1, \boldsymbol{J}_2, \dots, \boldsymbol{J}_n) \cdot \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{pmatrix} = \boldsymbol{J}_1 \cdot \dot{\theta}_1 + \boldsymbol{J}_2 \cdot \dot{\theta}_2 + \dots + \boldsymbol{J}_n \cdot \dot{\theta}_n$$

Kinematic Singularities: Jacobian Matrix (2)

The end effector velocity is a **linear combination** of the columns of the Jacobian matrix.

$$\dot{\boldsymbol{x}} = \boldsymbol{J}_1 \cdot \dot{\theta}_1 + \boldsymbol{J}_2 \cdot \dot{\theta}_2 + \cdots + \boldsymbol{J}_n \cdot \dot{\theta}_n \qquad J(\boldsymbol{\theta}) = (\boldsymbol{J}_1, \boldsymbol{J}_2 \dots, \boldsymbol{J}_n)$$

If a robot is in a configuration $\boldsymbol{\theta}_{singular} \in C$ in which it is no longer able to move instantaneously in one or more directions, this is referred to as a **kinematic singularity**.

In mathematical terms, kinematic singularity means that the linear combination of Jacobian columns **does not span the entire end effector velocity space**.

The Jacobian matrix $J(\theta)$ has a rank smaller than the workspace dimension.

$$rank\, J(\boldsymbol{\theta}) < m, \qquad \dot{\boldsymbol{x}} \in \mathbb{R}^m$$

Kinematic Singularities: Definition

Given a forward kinematics function f

$$\boldsymbol{x} = f(\boldsymbol{\theta}), \qquad \boldsymbol{\theta} \in C \subset \mathbb{R}^n, \qquad \boldsymbol{x} \in W \subset \mathbb{R}^m$$

and the corresponding Jacobian matrix

$$J(\boldsymbol{\theta}) = \left(\frac{\partial f}{\partial \theta_1}, \quad \frac{\partial f}{\partial \theta_2}, \quad \dots, \quad \frac{\partial f}{\partial \theta_n}\right) \in \mathbb{R}^{m \times n},$$

a configuration $\boldsymbol{\theta}_{\text{singular}} \in C$ is called **singular** if the rank of the Jacobian matrix is smaller than the dimension of the workspace.

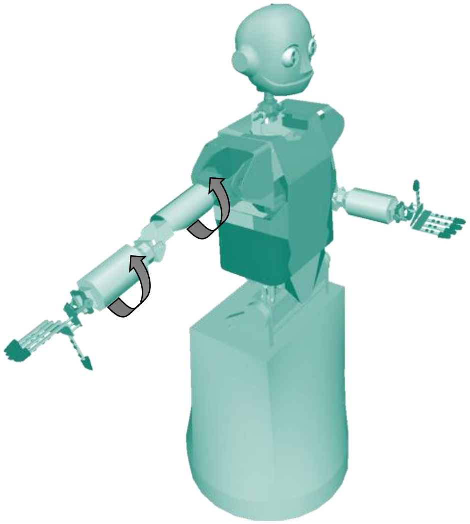
$$\operatorname{rank} J(\boldsymbol{\theta}) < m$$

Singularities

- **Definition**:
  A kinematic chain is in a singular configuration if the associated Jacobian matrix is not of full rank, i.e. two or more columns of $J(\boldsymbol{\theta})$ are linearly dependent.

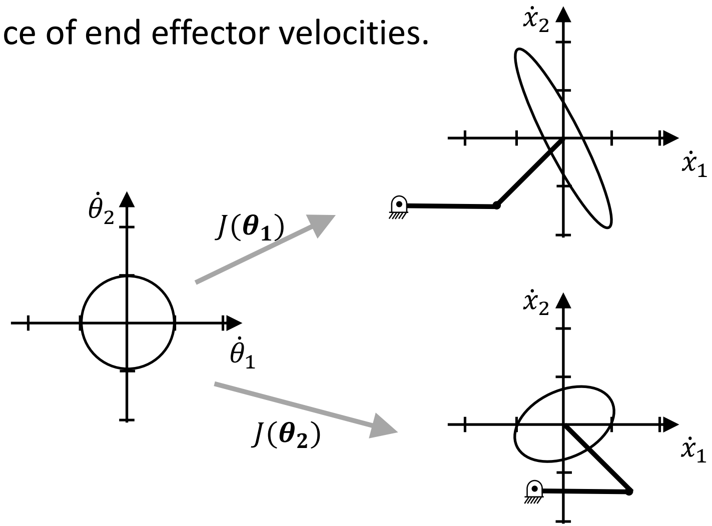
- A singular Jacobian matrix cannot be inverted
  $\Rightarrow$ Certain end effector movements are impossible

- In the vicinity of singularities, large joint velocities may be necessary to maintain an end effector velocity.



Manipulability

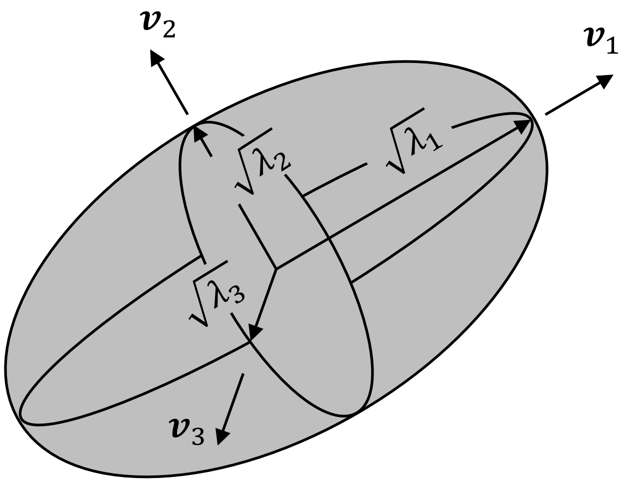
- **Manipulability** is aa **measure** of the freedom of movement of the end effector; indicates also how ‘close‘ a configuration is to a singularity
- **Manipulability ellipsoid**
  - Describes the end effector velocities for joint angle velocities with $\|\dot{\theta}\| = 1$
  - Use $J(\boldsymbol{\theta})$ to map the **unit circle** of joint angle velocities to the space of end effector velocities.
  - Result: **Manipulability ellipsoid**
  - Depends on joint angle configuration
- Analysis
  - **Circle (‘large ellipsoid‘):**
    End effector movement is possible without restriction in any direction.
  - **Degenerate cases (compressed ellipsoid):**
    End effector movement is restricted in certain directions.



Manipulability: Eigenvalue Analysis

- Calculate $A(\boldsymbol{\theta}) = J(\boldsymbol{\theta}) \cdot J(\boldsymbol{\theta})^T \in R^{\mathrm{m} \times m}$
- $A(\boldsymbol{\theta})$ is
  - Quadratic
  - Symmetric
  - Positive definite
  - Invertible
- Eigenvalues λ_i and Eigenvectors $\boldsymbol{v_i}$ of A
  - $A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$
  - $(\lambda_i I - A)\boldsymbol{v}_i = \boldsymbol{0}$
- Singular values
  - $\sigma_i = \sqrt{\lambda_i}$

Volume V is proportional to
$$\sqrt{\lambda_1 \lambda_2 \dots \lambda_m} = \sqrt{\det(A)} = \sqrt{\det(JJ^T)}$$



Manipulability ellipsoid:
Geometric representation of the manipulability

Manipulability Measure

- **Scalar** measures for manipulability
  - **Smallest singular value**

$$\mu_1(\theta) = \sigma_{min}\big(A(\theta)\big)$$

  - **Inverse condition**

$$\mu_2(\theta) = \frac{\sigma_{min}(A(\theta))}{\sigma_{max}(A(\theta))}$$

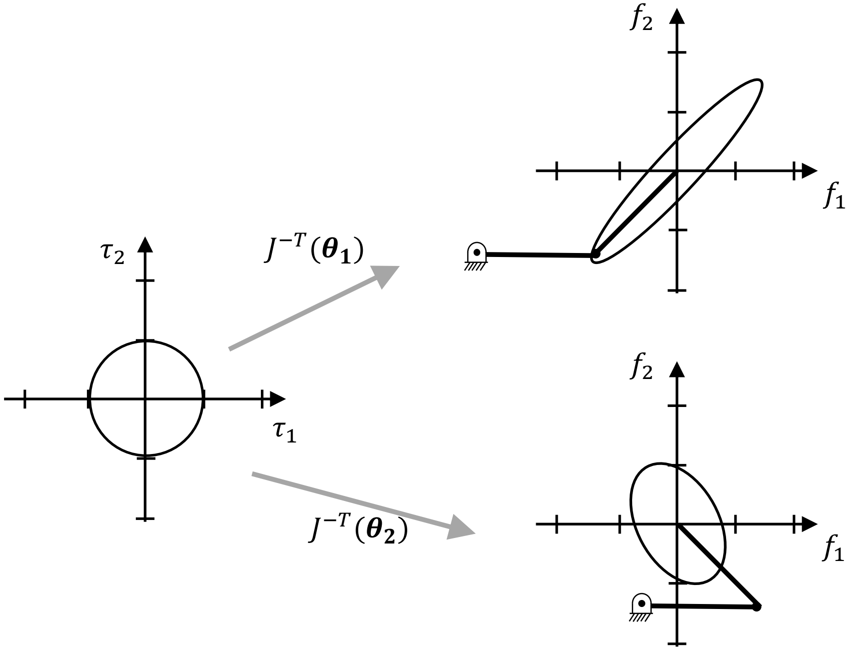
  - **Determinant**

$$\mu_3(\theta) = \det A(\theta)$$

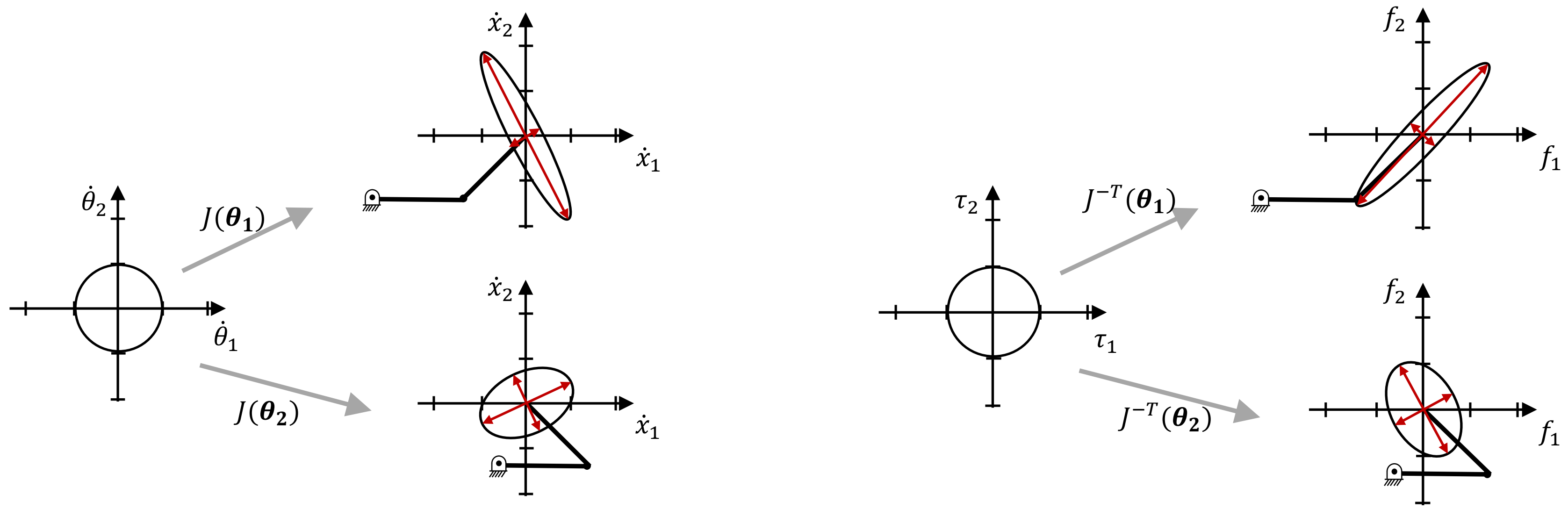
- Application:
  - Analysis of joint angle configurations
  - Singularity avoidance

Force Ellipsoid

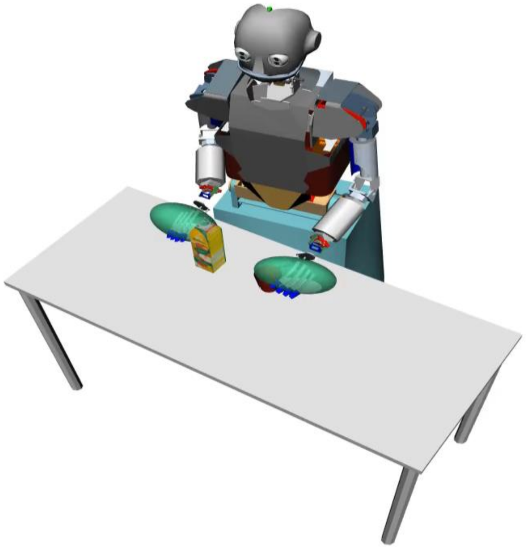
$$\tau(t) = J_f^T\big(\theta(t)\big) \cdot F(t) \qquad \rightarrow \qquad F(t) = J_f^{-T}\big(\theta(t)\big) \cdot \tau(t)$$



Manipulability and Force Ellipsoid



Manipulability – Examples



Contents

Joint Angle Limits

- A robot with the configuration space $C \subset \mathbb{R}^n$ generally only covers part of the underlying $\mathbb{R}^n$ as there are **joint angle limits**.
- There is a minimum and maximum value for each joint

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n) \in C$$

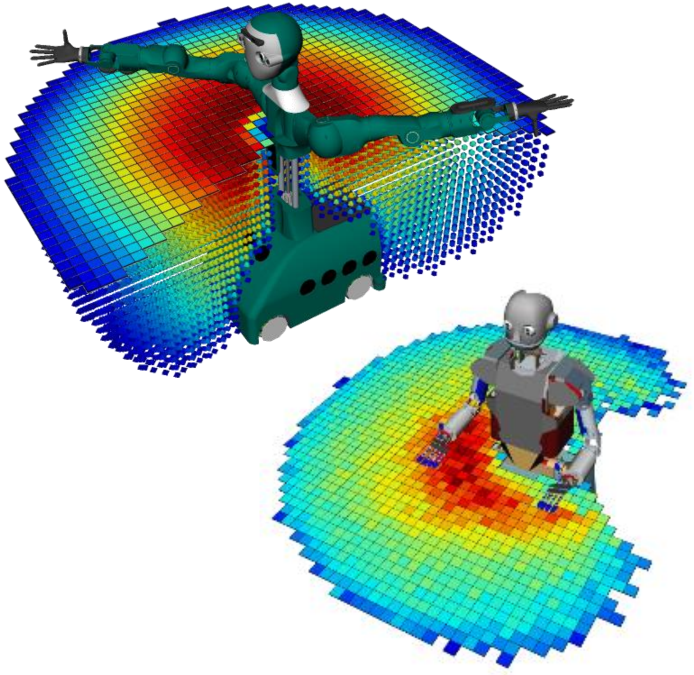
$$\theta_i \in \left[\theta_{i,\min}, \theta_{i,\max}\right]$$

- Exception: Continuous rotation joints (ARMAR-6)
- Joint angle limits restrict the **reachable part of the workspace**

$$W_{\text{reachable}} \subseteq W \subset \mathbb{R}^6$$

Representation of Reachability (1)

- Reachable part of the workspace of the robot in $\mathbb{R}^6$
- Approximation using a 6-dimensional grid
- Entry in each grid cell:
  - **Reachability:**
    Binary: Is there at least one joint angle configuration so that the Tool Center Point (TCP) lies within the 6D grid cell?
  - **Manipulability:**
    Maximum manipulability value of a grid cell, e.g. $\mu_1(\theta)$

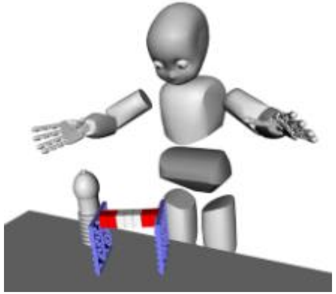


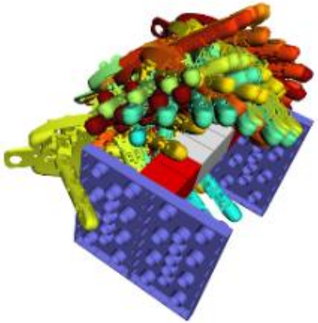
Visualization of reachability and manipulability for the ARMAR-6 and ARMAR-III robots

Vahrenkamp, N., Asfour, T. and Dillmann, R., *Efficient Inverse Kinematics Computation based on Reachability Analysis*, International Journal of Humanoid Robotics (IJHR), vol. 9, no. 4, 2012

Representation of Reachability (2)

- **Generation**
  - Offline process in simulation
  - Check all joint angles
    - in x steps (e.g. x = 5°)
    - Determine the pose of the TCP using forward kinematics
    - Determine the grid cell and set the entry
- **Application**
  - Pre-calculated reachability information for trajectory optimization
  - Quick decision whether a pose is reachable with the end effector.
    Constant computational complexity: $O(1)$
  - Can be used for grasp selection



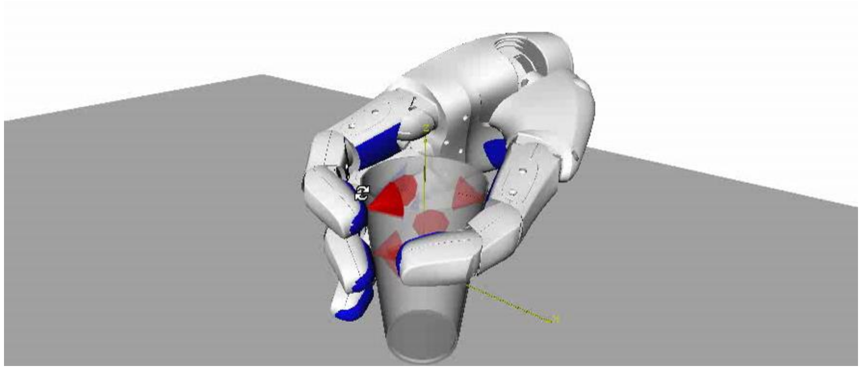


Grasps that cannot be reached can be efficiently sorted out.

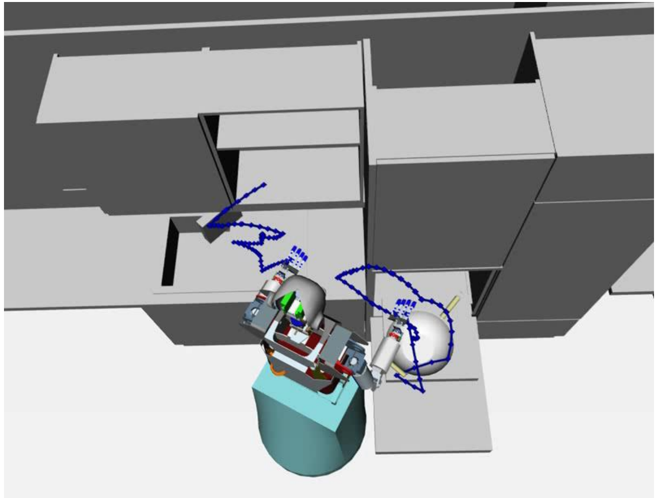
Contents

Geometric Model

- Collision, contact calculation and motion planning



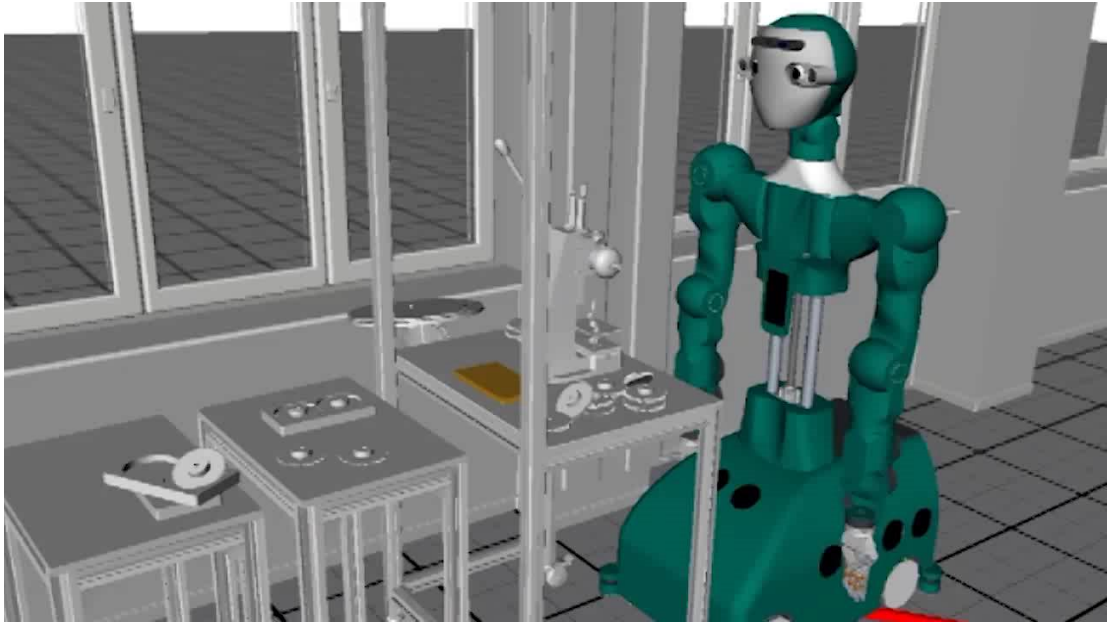
Grasping



Motion planning

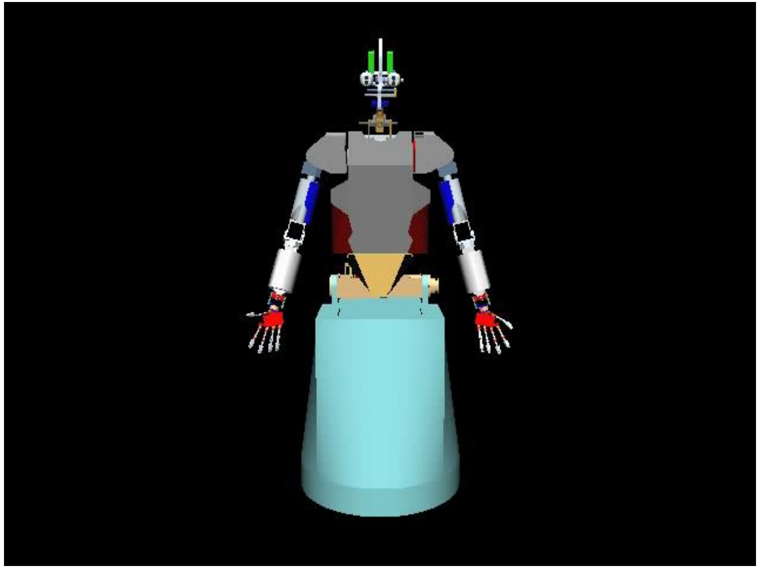
Geometric Model

- Collision freee motion planning

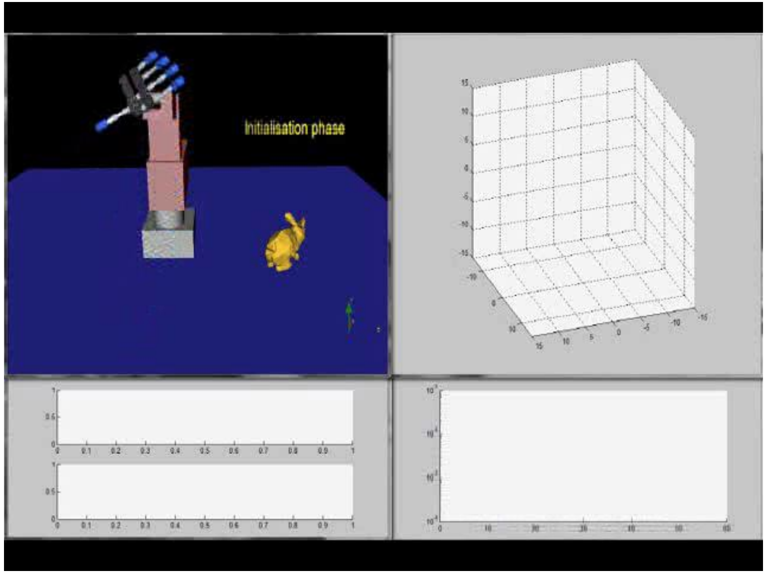


Geometric Model

- Simulation



Imitation

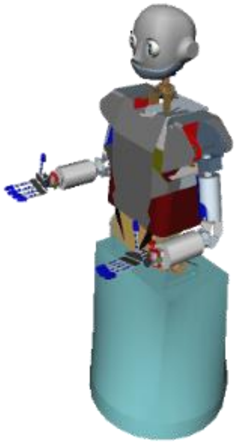


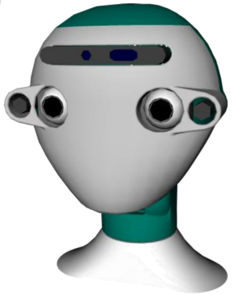
Haptic exploration

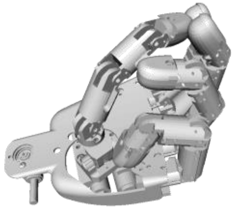
Geometric Model

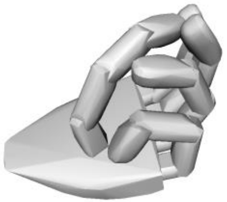
Application

- Graphical **representation** of bodies (visualization)
- Starting point for **distance calculation** and **collision detection**
- Basis for calculating the **movements** of body parts
- Basis for determining the acting **forces and torques**





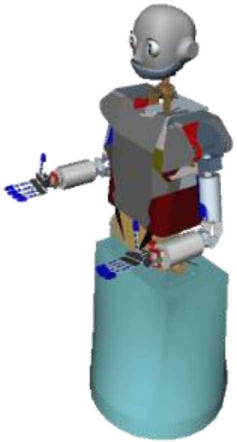
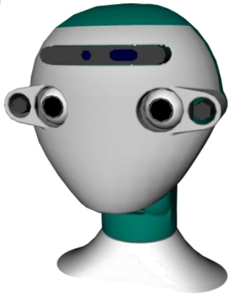


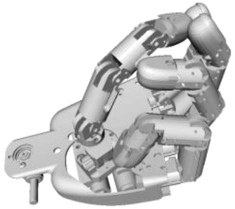
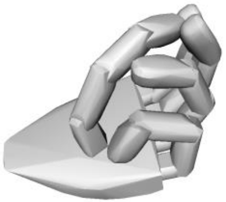


Geometric Model: Classification

- Classification according to **space**
  - 2D models
  - 3D models

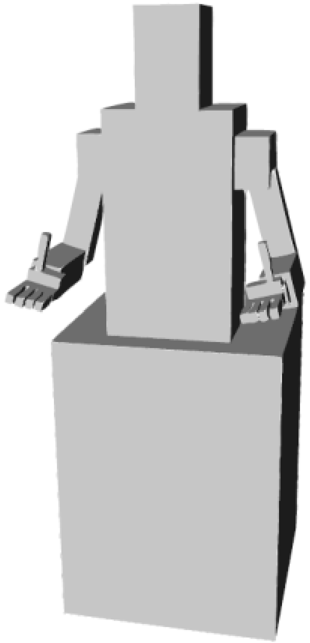
- Classification according to **basic primitives**
  - Edge or wireframe models
  - Surface models
  - Volume models

Block World

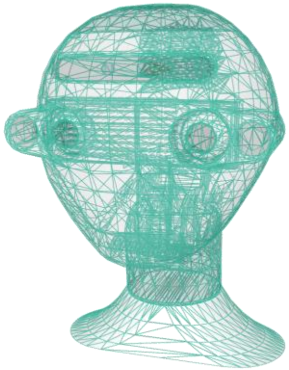
- The bodies are represented by **bounding boxes**.
- Used in the first steps of collision avoidance.
- **Class**: 3D, volumes or surfaces



ARMAR-III block world model

Edge Model

- The bodies are represented by **polygons** (edges).
- Used for quick visualization.
- **Class**: 3D, edges or surfaces

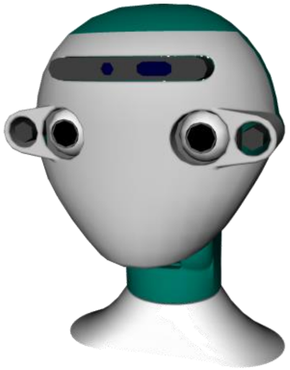


ARMAR-6 head model

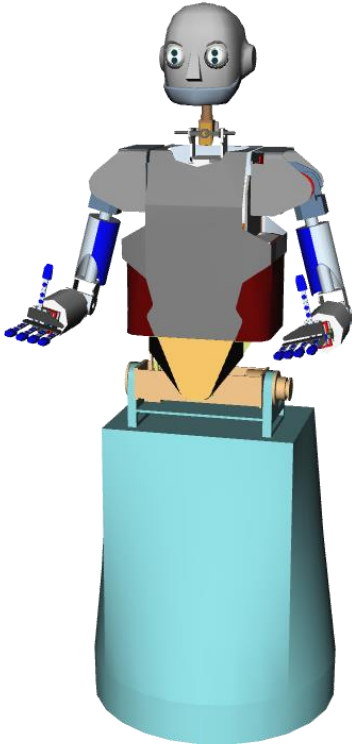
ARMAR-III edge model

Volume Model

- The bodies are represented **accurately**.
- Precise collision detection possible.
- Used for animations.
- **Class**: 3D, volume



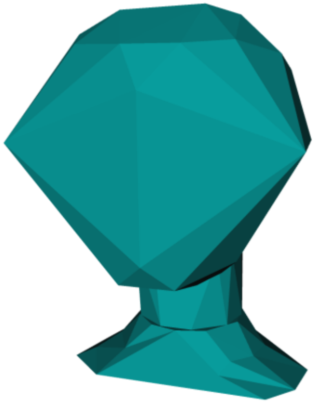
ARMAR-6 head model



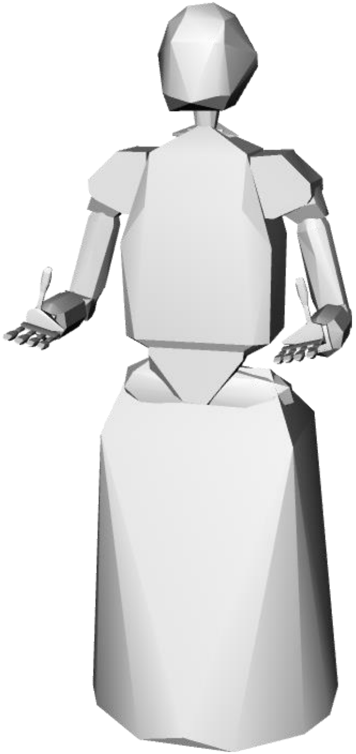
ARMAR-III volume model

Collision Model

- The bodies are represented in **simplified** form.
- Fast collision detection possible
- **Class**: 3D, volume



ARMAR-6 collision model



ARMAR-III collision model

Summary

- **Kinematic model**
  - Denavit-Hartenberg convention: Minimum number of parameters to describe transformation between consecutive joints
  - Direct kinematics problem: Calculate end-effector pose from joint angles
  - Jacobian matrix: The solution for everything ☺
  - Singularities and manipulability
  - Reachability

- **Geometric model**
  - Classification according to space (2D/3D) and basic primitives (edge or wireframe models, surface models and volume models)